How to Gamble If You Must (Expected Value)

1 Warm-Up: Carnival Games

1. You are at a carnival. Each game costs $1 to play.

   Pick a ducky. 70% of the duckies win nothing. 20% of the duckies win $2. 10% of the duckies win $5.
   Lollipop tree. 95% of lollipops win nothing, and 5% win $10.

   You decide to play the ducky game and your friend decides to play the lollipop game.

   (a) How much money should you expect to win, on average, on the ducky game?

   (b) How much money should your friend expect to win, on average, on the lollipop game?

   (c) If you pool your money, and play your respective games, how much should you expect to win or lose altogether?

2. In another carnival game, that also costs $1 to play, you pick a number between 1 and 52. Then you shuffle a deck of cards, and deal them in order, face up. If your number is 13, then you look at the 13th card. If it is an Ace, then you win $12; otherwise you win nothing.

   (a) What is the expected value for your winnings in this game? *(Expected value is what happens on average, if you play a lot of times.)*

   (b) If you and your friend play, and pick the same number, what is the expected value for your total winnings, pooled together?

   (c) What if you pick two consecutive numbers, say 1 and 2?

2 Expected Value is Additive

3. Place n letters at random into n envelopes. What is the average number of letters which get into the correct envelopes?

4. We randomly place 5 cats and 3 dogs in a row. On average, how many pairs of adjacent positions are "heterospecious", i.e. dog-cat or cat-dog?

5. (a) On average, how many times must a die be thrown until a six is rolled?

   (b) How many times, on average, must a die be thrown in order to see all 6 possible outcomes?

6. Shuffle an ordinary deck of 52 playing cards. On average, how far from the top will the first ace be?

7. An urn contains 1000 balls, labeled $1, 2, 3, \ldots, 1000$. You perform the following procedure: mix the balls well, pick a ball, record its number, and then put it back in the urn. If you do this procedure 1000 times, what is the average number of *distinct* integers that you will record?

3 Gambler's Ruin

8. *Finite Random Walks.* Suppose you are sitting on the number line, at position k, and you flip a fair coin. If it is heads, you move one space to the right. If it is tails, you move one space to the left. You lose if you hit 0 and win if you hit 10. What is the probability that you win?

   Generalize to other numbers n besides 10.

9. *The Classic Gambler's Ruin Problem.* Two players take turns tossing a fair coin. If the coin is heads, player A gives player B a dollar. If the coin turns up tails, B gives A a dollar. Player A starts with a dollars and player B starts with b dollars (a and b are non-negative integers). Once a player goes bankrupt (i.e. has zero dollars), the game is over. What is the probability that A goes bankrupt?

10. *A Gambling System.* Suppose you are playing a game with a 50% chance of winning each time. You can bet any amount, and if you win, you win twice your bet. If you lose, you lose your bet. In other words, if you bet B dollars, your *profit* is $\pm B$, depending if you win or lose. You decide that you will play, stopping as soon as you win, doubling the size of your bet each time. Are you guaranteed to make a profit? Try this strategy a few times, and see what happens. What is the expected value for your profit?

11. *The St. Petersburg Paradox.* Consider the following game. I will flip a fair coin until it shows up heads. We keep track of the number of flips until this happens. If it happens on the first flip, I'll pay you \$2. If it takes two flips, then I'll pay you \$4. Three flips, \$8, etc. In other words, if it takes n flips until the first head, I will pay you 2^n dollars. Pretty sweet game!

    How much is this game worth *to you*? In other words, if there was a ticket that allowed you to play the game once with me (I flip the coin until it is heads, and pay you the appropriate amount), how much would you pay for the ticket? Clearly, you'd play at least 1 dollar. In fact, you'd almost certainly pay 2 dollars. How about 3? 4? 5? More?