Lattice Polygon's and Pick's Theorem

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http://www.geometer.org/mathcircles/pick.pdf

1 Warm-Up to Ponder

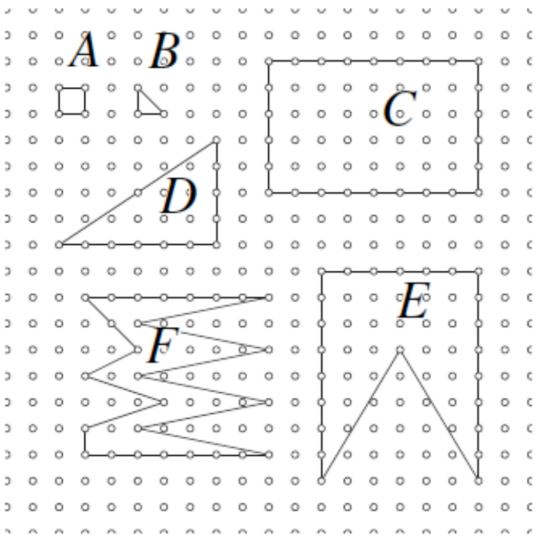
1. Is it possible to draw an equilateral triangle on graph paper so that all three vertices of the triangle lie on "lattice points" where the lines of the graph paper cross?
2. How many ways are there to make change for $2.00, using only nickels, dimes, and quarters?

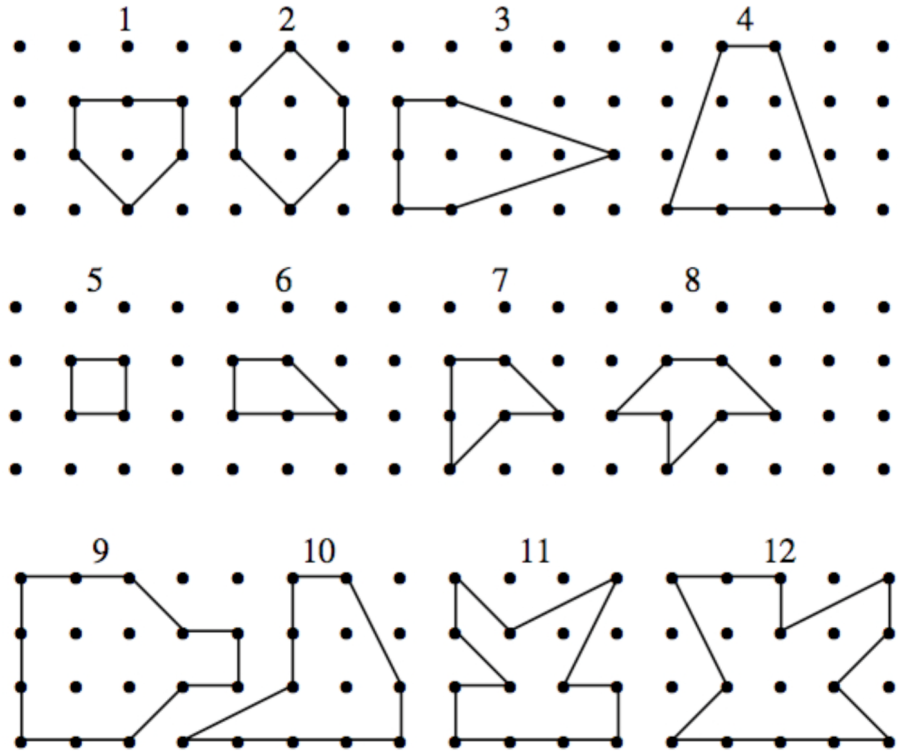
2 Lattice Polygons

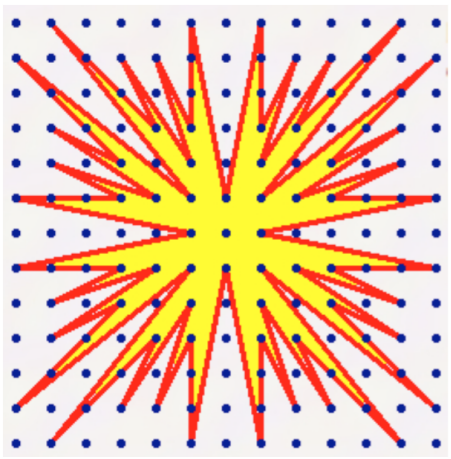
A *lattice point* in the plane is any point that has integer coordinates. A *lattice polygon* is a polygon whose vertices have integer coordinates.

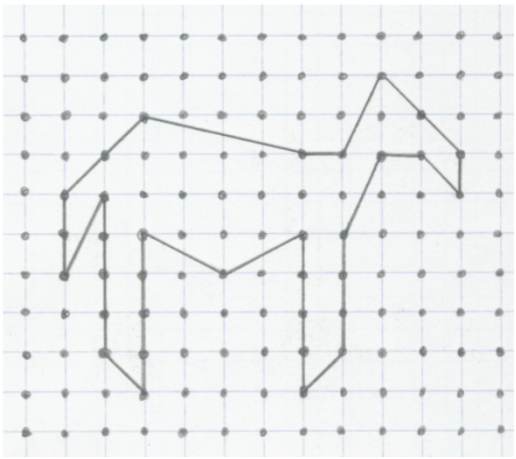
1. Find the area of each of the following lattice polygons below and on the next page. Make a table that contains the following information for each polygon: the area of the polygon, the number of lattice points inside the polygon (I), and the number of lattice points on the boundary of the polygon (B). Can you make any observations or conjectures?

Polygon	Area	I	B	Polygon	Area	I	B	Polygon	Area	I	B	Polygon	Area	I	B
A				1				7				sunburst			
B				2				8				horse			
C				3				9							
D				4				10							
E				5				11							
F				6				12							







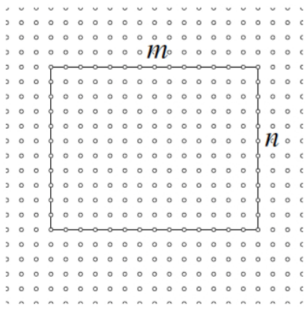


2. Based on your work above, can you make a conjecture about a formula that relates the area of a lattice polygon to the number of lattice points inside the polygon and the number of lattice points on the boundary of the polygon? This formula is known as Picks Theorem, and there are numerous beautiful proofs of this result. You can work through two of them using the outlines provided.

3 Proof of Pick's Theorem Using Induction

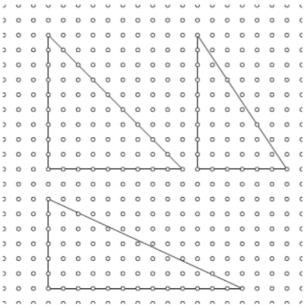
In this series of exercises, you will prove Picks Theorem using induction.

1. Consider an $m \times n$ lattice-aligned rectangle:

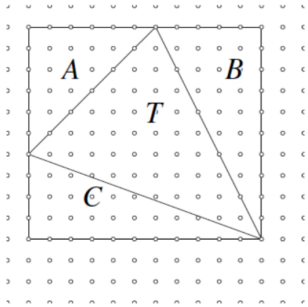


Show that for this rectangle, $I = (m-1)(n-1)$ and $B = 2m + 2n$. Conclude that $A = I + \frac{B}{2} - 1$.

2. Next, find I and B for a lattice-aligned right triangle with legs m and n. Prove that Pick's Theorem holds for such a triangle.



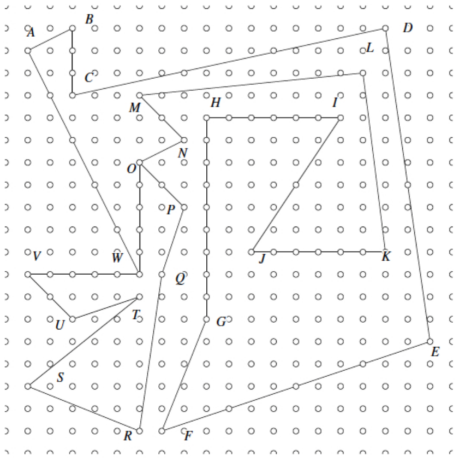
3. The next step is to show that Picks Theorem holds for arbitrary triangles. If T is an arbitrary triangle, draw right triangles A, B, C to form a rectangle R, as shown below.



Hint: Suppose that triangle A has I_a interior points and B_a boundary points. Use similar notation for triangles B and C. Let I_r and Br denote the number of interior and boundary points of the rectangle, respectively. First show that $B_r = B_a + B_b + B_c - 3$ and $I_r = I_a + I_b + I_c + I_t + (B_a + B_b + B_c - B_r) - 3$.

4. So far, we've shown that Picks Theorem is true for every polygon with 3 sides. To complete the proof that Picks Theorem is true for any polygon, we'll use induction on the number of sides of the polygon. The base case is $n = 3$ sides, and weve already shown that Picks Theorem holds for $n = 3$. For the inductive step, assume that Picks Theorem holds for $n = 3, 4, ..., k - 1$ sides. We must now prove that Picks Theorem holds for $n = k$ sides to complete the induction.

   Suppose that P is a polygon with k sides ($k > 3$). Show that P must have an interior diagonal that will split P into 2 smaller polygons. Heres an example. OW is the interior diagonal for this example.



5. Once we have shown that we can always split a polygon P with k sides into 2 smaller polygons P_1 and P_2 (each with fewer than k sides), the final step is to show that if 2 polygons satisfy Picks Theorem, then the polygon formed by attaching the 2 will also satisfy Picks Theorem. Since the smaller polygons satisfy Picks Theorem by the inductive hypothesis, we have

$$A(P) = A(P_1) + A(P_2) = I_1 + \frac{B_1}{2} - 1 + I_2 + \frac{B_2}{2} - 1$$

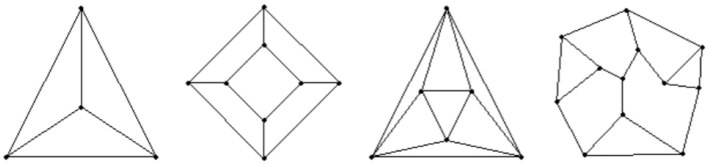
   Find a relationship between I and I_1 and I_2 and between B and B_1 and B_2 and use it to prove that

$$A(P) = I + \frac{B}{2} - 1$$

4 Proof of Pick's Theorem Using Euler's Formula

A graph $G = (V, E)$ consists of a finite set of vertices and a finite set of edges, where each edge connects two vertices (or a vertex with itself). Some examples of graphs are shown below. These graphs are both planar (can be drawn in such a way that no 2 edges cross) and connected (there is a path between every pair of distinct vertices of G). The "faces" of the graph are the regions of the plane surrounded by edges. Eulers Formula states that if G is a connected, planar graph with v vertices, e edges, and f faces, then$v-e+f = 2$.

For example, the graph on the far left below contains 4 vertices, 6 edges, and 4 faces, and $4-6+4 = 2$.



1. To prove Picks Theorem from Eulers Formula, start with a lattice polygon P, and show that you can always dissect the polygon into primitive lattice triangles. A primitive lattice triangle is a triangle that has no lattice points in its interior, and no lattice points other than vertices on its sides.

2. Next, show that the area of a primitive lattice triangle is equal to 1. (Of course, you cant use Picks Theorem since that's what were trying to prove! You must use geometry to prove this result.)

3. Once youve proved these results, it's possible to obtain Picks Theorem. Begin by observing that the vertices of the graph of P are the lattice points in the interior and on the sides of P, and the edges of the graph P are the sides of P and of the primitive triangles in the dissection of P . The f regions of the graph of P are the $f-1$ primitive triangles and the complement of P in the plane. Since each primitive triangle has area $\frac{1}{2}$, we have $A(P) = \frac{1}{2}f - 1$.

   Let e_i denote the number of interior edges of primitive triangles, and let e_s denote the number of edges of primitive triangles on the sides of P . Explain why

   $$3(f-1) = 2e_i + e_s = 2e - e_s$$

   Use Euler's Formula to rewrite this equation as

   $$f - 1 = 2(v-2) - e_s + 2$$

   Finally, use

   $$A(P) = \frac{1}{2}(f-1)$$

   and

   $$v = B(P) + I(P)$$

   to obtain Pick's Theorem.

5 Problem Solving Using Pick's Theorem

1. Is it possible to construct an equilateral lattice triangle? Why or why not?

2. How many ways are there to make change for \$2.00, using only nickels, dimes, and quarters? Express this problem in terms of a lattice polygon and then solve it.

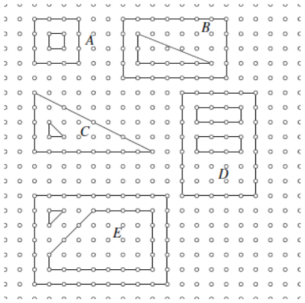
3. Find the equation of the line connecting 2 points $A(n, 0)$ and $B(0, n)$, and show that this line contains all points of the form $(i, n - i)$, where i is an integer. There are $n - 1$ such points between A and B. Connect each one of them with the origin $O(0, 0)$. The lines divide $\triangle OAB$ into n small triangles. It is clear that the 2 triangles next to the axes (i.e. the triangle adjacent to the x-axis and the triangle adjacent to the y-axis) contain no lattice points in their interior. Prove that if n is prime, then each of the remaining triangles contains exactly the same number of interior lattice points. Find an expression (in terms of n) for the number of interior lattice points in each of these triangles.

4. (1987 IMO) Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any 2 points is irrational and each set of three points determines a non-degenerate triangle with rational area.

5. An $n \times n$ square is randomly tossed onto the plane. Prove that it may never contain more than $(n + 1)^2$ lattice points.

6. Suppose that T is a lattice triangle and $I(T) = 0$. What are the possible values for $B(T)$? Can you find an example for each possible value?

7. If T is a lattice triangle with $I(T) = 1$, what are the possible values for $B(T)$? Can you find an example for each possible value?

8. Is it possible to construct an equilateral lattice square? A regular lattice hexagon? For which n is it possible to construct a regular lattice n-gon (i.e. a convex polygon that is equilateral and equiangular)?

9. Polygons with Holes. In the following figure, there are 5 examples of polygons with holes. Polygons A, B, C have one hole, and polygons D and E have 2 holes. Find the area of each of these polygons. Make a table that contains the following information for each polygon: I, B, area, number of holes. Doing more examples if necessary, modify Picks Theorem to derive a formula that works for polygons with holes. Then develop a proof of your conjecture.