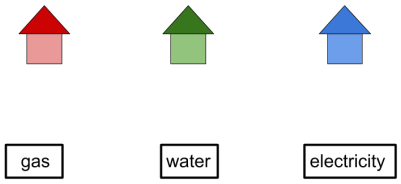
Euler Characteristic

1 Warm-up Problems

1. Suppose there are three cottages on a plane and each needs to be connected to the gas, water, and electricity companies. Using a third dimension or sending any of the connections through another company or cottage is not allowed. Is there a way to make all nine connections without any of the lines crossing each other?



2. Is it possible to place four points on a sheet of paper and connect every pair of points with a straight line segment, so that none of the segments intersect?
3. Is it possible to place five points on a sheet of paper and connect every pair of points with a straight line segment, so that none of the segments intersect?

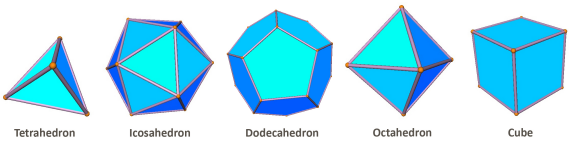
2 Polyhedra

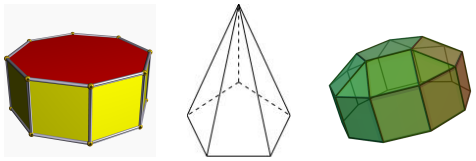
A polyhedron is a 3-dimensional shape with flat sides and straight edges. We call the sides "faces" and we call the points where edges meet the "vertices".

4. What are some examples of polyhedra?
5. Is it possible to build a polyhedron whose faces are
   (a) 8 triangles and 4 squares?
   (b) 4 triangles and 4 squares?
   (c) 4 triangles and 2 squares?

6. Complete this chart to take inventory of the number of faces, edges, and vertices in some popular polyhedra.

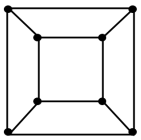
(Semi) regular	# of faces (F)	# of edges (E)	# of vertices (V)
Cube			
Tetrahedron			
Octahedron			
Icosohedron			
Dodecahedron			
Prism on an n-sided base			
Pyramid on an n-sided base			
Pentagonal cupola			





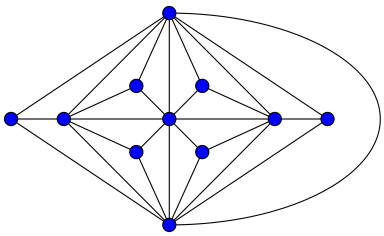
7. What patterns do you notice in the number of faces, edges, and vertices?

8. One way to represent a polyhedron on a flat piece of paper is as follows. Imagine the polyhedra is drawn on a flexible rubber ball. Take any face and punch a hole in it, then stretch the edges of that hole until the hole is much bigger than the original polyhedra. For example, this would turn a cube into the following figure. The punctured face is now the infinite outside region of the figure.

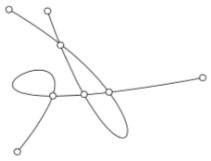


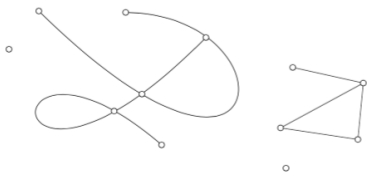
Represent a tetrahedron and your other favorite polyhedra in a similar way. We won't worry about straight edges anymore.

9. Experiment with drawing other *planar graphs*. Can you find any that DON'T follow the patterns you found in the chart? (A planar graph is a collection of vertices and edges drawn in the plane, in such a way that there is a vertex at the beginning and end of each edge and at any place where two edges meet or cross.)



10. What patterns hold for the number of faces, edges, and vertices for these planar graphs (also known as *scribbles*)?





11. What happens to the pattern you have observed if you remove an edge from a scribble?

3 Problems involving Euler Characteristic

12. In a certain small country there are villages, expressways, and fields. Expressways only lead from one village to another and do not cross one another, and it is possible to travel from any village to any other village along the expressways. Each field is completely enclosed by expressways and villages. If there are ten villages and sixteen expressways, then how many fields are there in this country?

13. A soccer ball is made up of pentagons and hexagons in such a way that three polygons meet at each vertex. How many pentagons must there be? Prove that no other number of pentagons is possible.

14. A certain polyhedron is built entirely from triangles, in such a way that 5 faces meet at each vertex. Prove that it has to have 20 faces. (Hint: first deduce that $3F = 2E$ and $3F = 5V$)

15. A *Platonic solid* is a polyhedron whose faces are all the same regular polygon (for example, all equilateral triangles or all squares) and for which each vertex has the same number of faces meeting at it (for example, 3 faces meet at each vertex). For example, a cube is a Platonic solid because all its faces are squares and every vertex has exactly three squares around it. But if we glued two cubes together and looked at the polyhedron made from the exposed faces, it would not be a Platonic solid because some vertices would have 3 squares around them and some would have 4.

    How many platonic solids are there? Prove it! Hint: use the previous problem, and other cases like it.

16. Can you solve the utilities problem or prove that it cannot be done?

17. Can you solve the problem of drawing 5 vertices and an edge between each pair of vertices, in such a way that no edges cross?

Thanks to Circle in a Box by Sam Vandervelde and www.geometer.org by Tom Davis for some of these problems. Thanks to Wikimedia for the pictures.