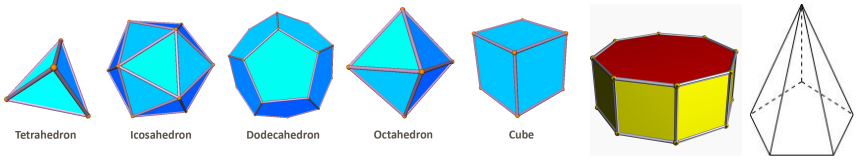
1 Polyhedra

A polyhedron is a 3-dimensional shape with flat sides and straight edges. We call the sides "faces" and we call the points where edges meet the "vertices".



For this lesson, we will only work with polyhedra that have "the topology of a sphere". In other words, polyhedra that have holes in them, like donuts, will not be allowed.

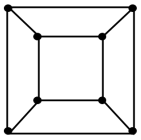
1. Is it possible to build a polyhedron whose faces are
   (a) 8 triangles?
   (b) 8 triangles and 4 squares?
   (c) 8 triangles and 2 squares?
   (d) 5 triangles?
2. Complete this chart to take inventory of the number of faces, edges, and vertices in some popular polyhedra.

(Semi) regular	# of faces (F)	# of edges (E)	# of vertices (V)
Cube			
Tetrahedron			
Octahedron			
Icosohedron			
Dodecahedron			
Prism on an n-sided base			
Pyramid on an n-sided base			

3. What pattern do you notice in the number of faces, edges, and vertices? This pattern is called Euler's formula.

2 Planar Graphs

4. One way to represent a polyhedron on a flat piece of paper is as follows. Imagine the polyhedra is drawn on a flexible rubber ball. Take any face and punch a hole in it, then stretch the edges of that hole until the hole is much bigger than the original polyhedra. For example, this would turn a cube into the following figure. The punctured face is now the infinite outside region of the figure.

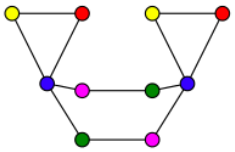


Represent a tetrahedron and your other favorite polyhedra in a similar way. We won't worry about straight edges anymore.

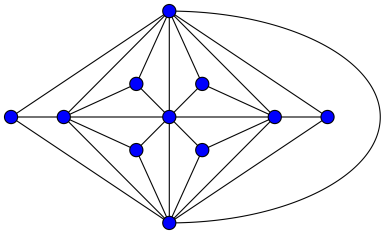
5. Experiment with drawing other *planar graphs*. A planar graph is a collection of vertices and edges drawn in the plane, in such a way that there is a vertex at the beginning and end of each edge and at any place where two edges meet or cross.

How many faces (F), edges (E), and vertices (V) are there in each of these planar graphs? Count the infinite outside region as one face. Does Euler's formula hold for these planar graphs?

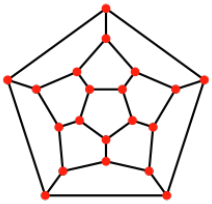
(a)



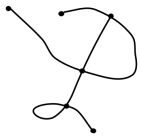
(b)

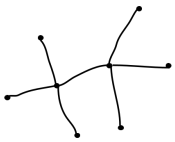


(c)

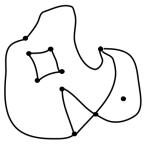


6. Count the number of faces (F), edges (E), and vertices (V) for these planar graphs. Does Euler's formula hold for all of them?

(a) 

(b) 

(c)

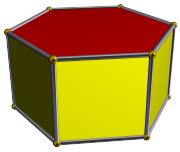
(d) 

7. Modify Euler's formula so that it holds for all of these planar graphs (also known as *scribbles*).

8. Prove that the pattern you found holds for all planar graphs with 0 edges. Prove it for planar graphs with 1 edge. Prove it for planar graphs with n edges, for any n.

3 Census of Polyhedra

9. This hexagonal prism has 8 faces and 12 vertices. Can you build a polyhedron with 12 faces and 8 vertices?



10. Verify that this truncated tetrahedron has 8 faces and 12 vertices. Can you find a polyhedron with 12 faces and 8 vertices?

11. If you can build a polyhedron with X faces and Y vertices, can you always also build one with Y faces and X vertices? Prove or give a counterexample.

12. The table below is a partial list of the number of vertices, edges, and faces (V, E, F) of some polyhedra. Fill in the missing entries.

V	E	F
8		6
6		8
4		6
6		4
5	8	
6		6
7	15	
	15	10
12	30	
24	36	

13. For each row of the table, can you find some polyhedron that has the corresponding numbers V, E, F? For example, the first row might be a cube.

14. Suppose a polyhedron has 60 faces, all of them triangles. What are V, E, and F?

15. The following table gives the number of distinct polyhedra for each number of vertices and faces.

Number of Polyhedra														
Total (row)	*Vertex Count* in row and *Face Count* in column													
96262938	**13**									219	7916	104213	709302	2937495
6384634	**12**								14	558	8822	64439	268394	709302
440564	**11**								38	768	6134	25626	64439	104213
32300	**10**							5	76	633	2635	6134	8822	7916
2606	**9**							8	74	296	633	768	558	219
257	**8**						2	11	42	74	76	38	14	
34	**7**						2	8	11	8	5			
7	**6**					1	2	2	2					
2	**5**					1	1							
1	**4**				1									
0	**3**													
0	**2**													
0	**1**													
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**

From http://www.numericana.com/data/polyhedra.htm.

(a) Why is the table symmetric across the line $y = x$?

(b) Why do all the non-zero entries lie in a wedge-shaped region? What are the slopes of the lines that bound the wedge? Hint: if there are F faces, what is the smallest number of edges there can be?

4 Problems involving Euler Characteristic

16. In a certain small country there are villages, expressways, and fields. Expressways only lead from one village to another and do not cross one another, and it is possible to travel from any village to any other village along the expressways. Each field is completely enclosed by expressways and villages. If there are ten villages and sixteen expressways, then how many fields are there in this country? [1]

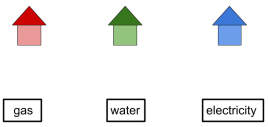
17. A soccer ball is made up of pentagons and hexagons in such a way that three polygons meet at each vertex. How many pentagons must there be? Prove that no other number of pentagons is possible.

18. A certain polyhedron is built entirely from triangles, in such a way that 5 faces meet at each vertex. Prove that it has to have 20 faces. (Hint: first deduce that $3F = 2E$ and $3F = 5V$)

19. A *Platonic solid* is a polyhedron whose faces are all the same regular polygon (for example, all equilateral triangles or all squares) and for which each vertex has the same number of faces meeting at it (for example, 3 faces meet at each vertex). For example, a cube is a Platonic solid because all its faces are squares and every vertex has exactly three squares around it. But if we glued two cubes together and looked at the polyhedron made from the exposed faces, it would not be a Platonic solid because some vertices would have 3 squares around them and some would have 4.

    How many platonic solids are there? Prove it! Hint: use the previous problem, and other cases like it.

20. Suppose there are three cottages on a plane and each needs to be connected to the gas, water, and electricity companies. Using a third dimension or sending any of the connections through another company or cottage is not allowed. Is there a way to make all nine connections without any of the lines crossing each other?



21. Is it possible to place four points on a sheet of paper and connect every pair of points with a straight line segment, so that none of the segments intersect?

22. Is it possible to place five points on a sheet of paper and connect every pair of points with a straight line segment, so that none of the segments intersect?

[1] Thanks to Circle in a Box by Sam Vandervelde and www.geometer.org by Tom Davis for some of these problems. Thanks to Wikimedia for most of the pictures.