# Lattice Polygon's and Pick's Theorem

**From Dana Paquin and Tom Davis**

http://www.geometer.org/mathcircles/pick.pdf

## 1 Warm-Up to Ponder

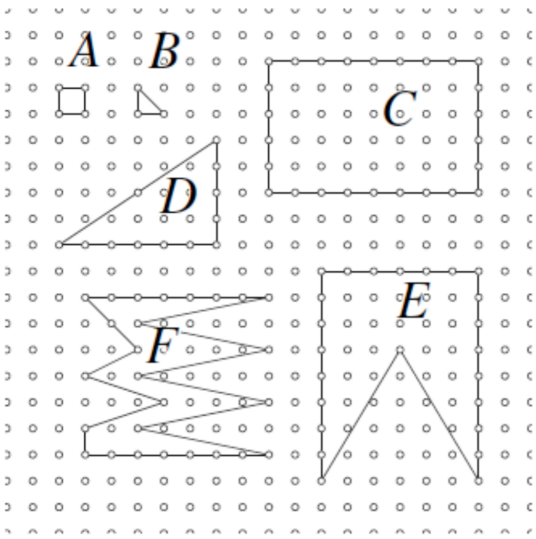
1. Is it possible to draw an equilateral triangle on graph paper so that all three vertices of the triangle lie on "lattice points" where the lines of the graph paper cross?
2. How many ways are there to make change for $2.00, using only nickels, dimes, and quarters?

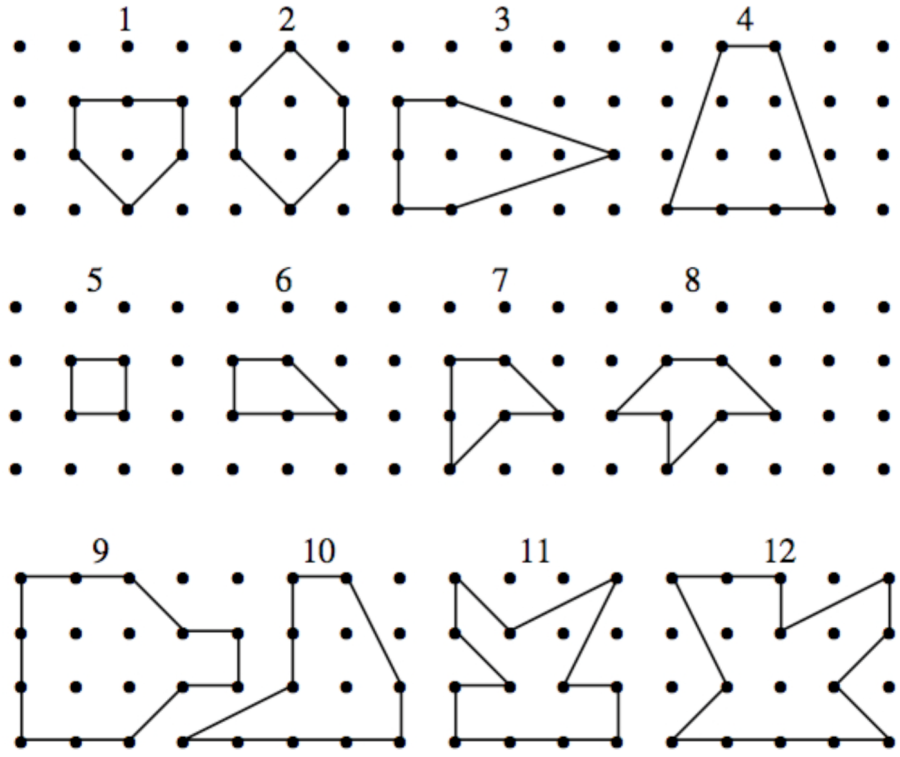
## 2 Lattice Polygons

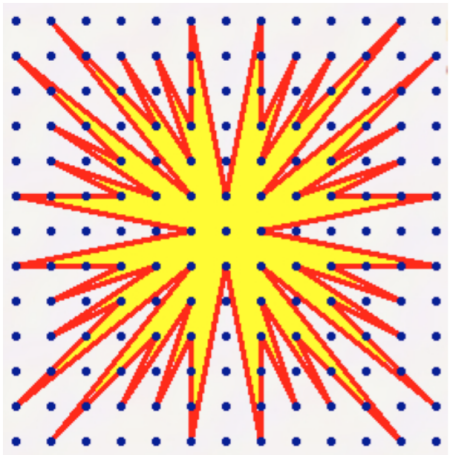
A *lattice point* in the plane is any point that has integer coordinates. A *lattice polygon* is a polygon whose vertices have integer coordinates.

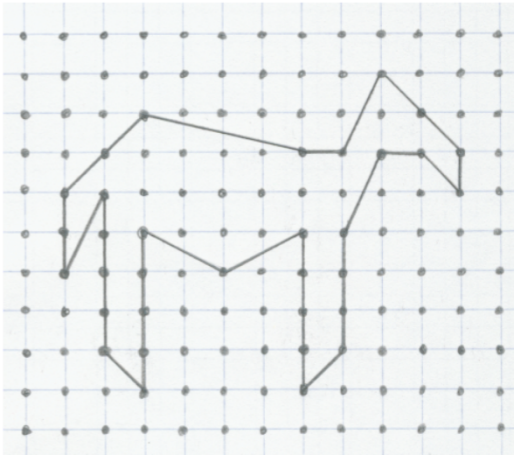
1. Find the area of each of the following lattice polygons below and on the next page. Make a table that contains the following information for each polygon: the area of the polygon, the number of lattice points inside the polygon (I), and the number of lattice points on the boundary of the polygon (B). Can you make any observations or conjectures?

| Polygon | Area | $I$ | $B$ | Polygon | Area | $I$ | $B$ | Polygon | Area | $I$ | $B$ | Polygon | Area | $I$ | $B$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | | | | 1 | | | | 7 | | | | sunburst | | | |
| B | | | | 2 | | | | 8 | | | | horse | | | |
| C | | | | 3 | | | | 9 | | | | | | | |
| D | | | | 4 | | | | 10 | | | | | | | |
| E | | | | 5 | | | | 11 | | | | | | | |
| F | | | | 6 | | | | 12 | | | | | | | |





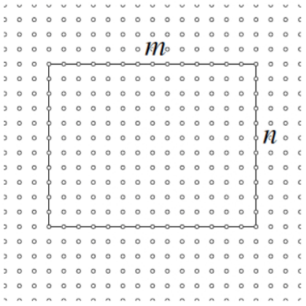


2. Based on your work above, can you make a conjecture about a formula that relates the area of a lattice polygon to the number of lattice points inside the polygon and the number of lattice points on the boundary of the polygon? This formula is known as Pick's Theorem, and there are numerous beautiful proofs of this result. You can work through one or both of them using the outlines provided.

# 3 Proof of Pick's Theorem Using Induction

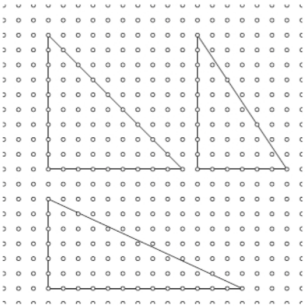
In this series of exercises, you will prove Pick's Theorem using induction.

1. Consider an $m \times n$ lattice-aligned rectangle:

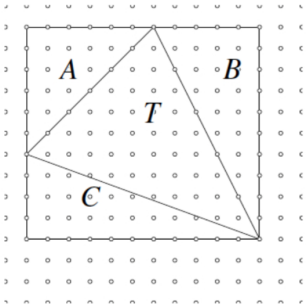



Show that for this rectangle, $I = (m-1)(n-1)$ and $B = 2m + 2n$. Conclude that $A = I + \frac{B}{2} - 1$.

2. Next, find $I + \frac{B}{2}$ for a lattice-aligned right triangle with legs $m$ and $n$. Prove that Pick's Theorem holds for such a triangle.



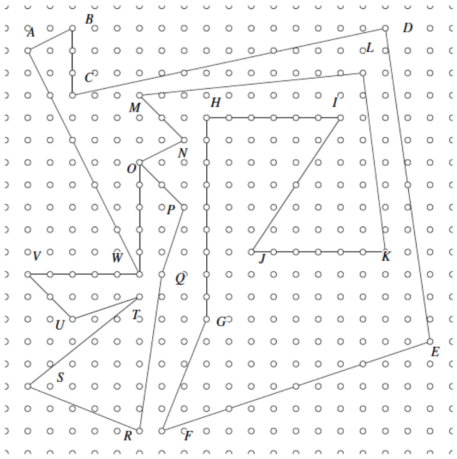
3. The next step is to show that Pick's Theorem holds for arbitrary triangles. If $T$ is an arbitrary triangle, draw right triangles $A, B, C$ to form a rectangle $R$, as shown below.




Hint: Suppose that triangle $A$ has $I_a$ interior points and $B_a$ boundary points. Use similar notation for triangles $B$ and $C$. Let $I_r$ and Br denote the number of interior and boundary points of the rectangle, respectively. First show that $B_r = B_a + B_b + B_c - 3$ and $I_r = I_a + I_b + I_c + I_t + (B_a + B_b + B_c - B_r) - 3$.

4. So far, we've shown that Pick's Theorem is true for every polygon with 3 sides. To complete the proof that Pick's Theorem is true for any polygon, we'll use induction on the number of sides of the polygon. The base case is $n = 3$ sides, and we've already shown that Pick's Theorem holds for $n = 3$. For the inductive step, assume that Pick's Theorem holds for $n = 3, 4, ..., k - 1$ sides. We must now prove that Pick's Theorem holds for $n = k$ sides to complete the induction.

   Suppose that $P$ is a polygon with $k$ sides ($k > 3$). Show that $P$ must have an interior diagonal that will split $P$ into 2 smaller polygons. Here's an example. $OW$ is the interior diagonal for this example.




5. Once we have shown that we can always split a polygon P with k sides into 2 smaller polygons $P_1$ and $P_2$ (each with fewer than $k$ sides), the final step is to show that if 2 polygons satisfy Pick's Theorem, then the polygon formed by attaching the 2 will also satisfy Pick's Theorem. Since the smaller polygons satisfy Pick's Theorem by the inductive hypothesis, we have

$$A(P) = A(P_1) + A(P_2) = I_1 + \frac{B_1}{2} - 1 + I_2 + \frac{B_2}{2} - 1$$

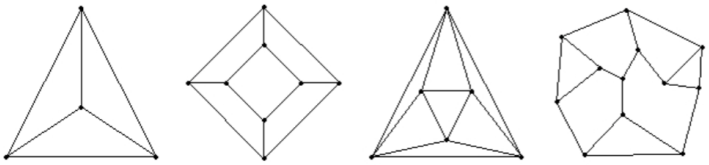
   Find a relationship between $I$ and $I_1$ and $I_2$ and between $B$ and $B_1$ and $B_2$ and use it to prove that

$$A(P) = I + \frac{B}{2} - 1$$

# 4 Proof of Pick's Theorem Using Euler's Formula

A graph $G = (V, E)$ consists of a finite set of vertices and a finite set of edges, where each edge connects two vertices (or a vertex with itself). Some examples of graphs are shown below. These graphs are both planar (can be drawn in such a way that no 2 edges cross) and connected (there is a path between every pair of distinct vertices of $G$). The "faces" of the graph are the regions of the plane surrounded by edges, including the unbounded region in the exterior. Euler's Formula states that if $G$ is a connected, planar graph with $v$ vertices, $e$ edges, and $f$ faces, then $v - e + f = 2$.

For example, the graph on the far left below contains 4 vertices, 6 edges, and 4 faces, and $4 - 6 + 4 = 2$.



1. To prove Pick's Theorem from Euler's Formula, start with a lattice polygon $P$, and show that you can always dissect the polygon into primitive lattice triangles. A primitive lattice triangle is a triangle that has no lattice points in its interior, and no lattice points other than vertices on its sides.

2. Next, show that the area of a primitive lattice triangle is equal to 1. (Of course, you can't use Pick's Theorem since that's what we're trying to prove! You must use geometry to prove this result.)

3. Once you've proved these results, it's possible to obtain Pick's Theorem. Begin by observing that the vertices of the graph of $P$ are the lattice points in the interior and on the sides of $P$, and the edges of the graph $P$ are the sides of $P$ and of the primitive triangles in the dissection of $P$. The $f$ regions of the graph of $P$ are the $f - 1$ primitive triangles and the complement of $P$ in the plane. Since each primitive triangle has area $\frac{1}{2}$, we have $A(P) = \frac{1}{2}f - 1$.

   Let $e_i$ denote the number of interior edges of primitive triangles, and let $e_s$ denote the number of edges of primitive triangles on the sides of $P$ . Explain why

   $$3(f - 1) = 2e_i + e_s = 2e - e_s$$

   Use Euler's Formula to rewrite this equation as

   $$f - 1 = 2(v - 2) - e_s + 2$$

   Finally, use

   $$A(P) = \frac{1}{2}(f - 1)$$

   and

   $$v = B(P) + I(P)$$

   to obtain Pick's Theorem.

# 5 Problem Solving Using Pick's Theorem

1. Is it possible to construct an equilateral lattice triangle? Why or why not?

2. How many ways are there to make change for \$2.00, using only nickels, dimes, and quarters? Express this problem in terms of a lattice polygon and then solve it.

3. Find the equation of the line connecting 2 points $A(n, 0)$ and $B(0, n)$, and show that this line contains all points of the form $(i, n - i)$, where $i$ is an integer. There are $n - 1$ such points between $A$ and $B$. Connect each one of them with the origin $O(0, 0)$. The lines divide $\triangle OAB$ into $n$ small triangles. It is clear that the 2 triangles next to the axes (i.e. the triangle adjacent to the $x$-axis and the triangle adjacent to the $y$-axis) contain no lattice points in their interior. Prove that if $n$ is prime, then each of the remaining triangles contains exactly the same number of interior lattice points. Find an expression (in terms of $n$) for the number of interior lattice points in each of these triangles.

4. (1987 IMO) Let $n$ be an integer greater than or equal to 3. Prove that there is a set of $n$ points in the plane such that the distance between any 2 points is irrational and each set of three points determines a non-degenerate triangle with rational area.

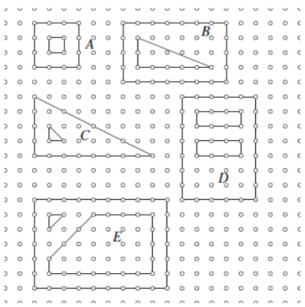
5. An $n \times n$ square is randomly tossed onto the plane. Prove that it may never contain more than $(n + 1)^2$ lattice points.

6. Suppose that $T$ is a lattice triangle and $I(T) = 0$. What are the possible values for $B(T)$? Can you find an example for each possible value?

7. If $T$ is a lattice triangle with $I(T) = 1$, what are the possible values for $B(T)$? Can you find an example for each possible value?

8. Is it possible to construct an equilateral lattice square? A regular lattice hexagon? For which n is it possible to construct a regular lattice n-gon (i.e. a convex polygon that is equilateral and equiangular)?

9. Polygons with Holes. In the following figure, there are 5 examples of polygons with holes. Polygons $A, B, C$ have one hole, and polygons $D$ and $E$ have 2 holes. Find the area of each of these polygons. Make a table that contains the following information for each polygon: $I$, $B$, area, number of holes. Doing more examples if necessary, modify Pick's Theorem to derive a formula that works for polygons with holes. Then develop a proof of your conjecture.



# 6 Application of Pick's Theorem to Continued Fractions

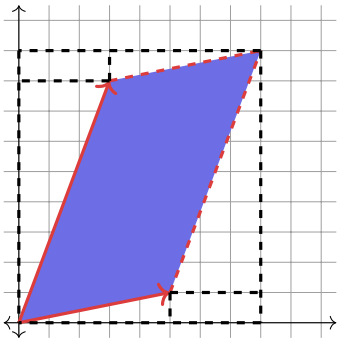
**Proposition 6.1.** *Let $[a_0; a_1, a_2, \ldots, a_n]$ be a continued fraction. Prove that for each integer $k$ with $1 \leq k \leq n$, we have*

$$\det \begin{bmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{bmatrix} := P_{k-1}Q_k - Q_{k-1}P_k = (-1)^k.$$

*Remark.* Proposition 6.1 is Exercise #2.2 of the CHMC worksheet for the advanced group of October 16, 2021.

**Proposition 6.2.** *If $\vec{v}_1 := (a, b), \vec{v}_2 := (c, d) \in \mathbf{R}^2$, then the area enclosed by the parallelogram determined by $\vec{v}_1$ and $\vec{v}_2$ is*

$$|\det(\vec{v}_1, \vec{v}_2)| := \left|\det \begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = |ad - bc|\,.$$



**Definition 6.3.** A *finite simple continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}},$$

where $a_0$ is an integer, and $a_1, a_2, \ldots, a_n$ are all positive integers. To simplify this notation we let

$$[a_0; a_1, a_2, \ldots, a_n]$$

denote the above expression.

**Definition 6.4.** An *infinite simple continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}}},$$

where $a_0$ is an integer, and $a_1, a_2, \ldots, a_n, \ldots$ are all positive integers. As in Definition 6.3, we let

$$[a_0; a_1, a_2, \ldots, a_n, \ldots]$$

denote the above infinite simple continued fraction.

**Definition 6.5.** Let $\alpha$ be either a simple finite continued fraction or an infinite simple continued fraction as in the above definitions. Further, let $k$ be an integer with $0 \leq k \leq n$. Then the $k$th *convergent* to $\alpha$ is the finite continued fraction

$$\alpha_k := [a_0; a_1, \ldots, a_k].$$

*Notation:.* We write

$$\alpha_k = \frac{P_k}{Q_k},$$

where $P_k, Q_k \in \mathbf{Z}$, $Q_k \geq 1$, and $\gcd(P_k, Q_k) = 1$. That is, $\frac{P_k}{Q_k}$ is the lowest-terms fraction for the $k$th convergent to $\alpha$ with positive denominator.

**Proposition 6.6.** *Prove that for any continued fraction $[a_0; a_1, a_2, \ldots, a_n]$, we have $Q_0 \geq 1$, $Q_1 \geq 1$, $Q_2 \geq 2$, $Q_3 \geq 3$, $Q_4 \geq 5$, and in general, $Q_n > Q_{n-1}$ and $Q_n > n + 1$ for all $n \geq 3$.*

*Remark.* Proposition 6.6 appeared as Exercise #2.1 from the aforementioned worksheet from October 16, 2021.

**Theorem 6.7** (Pick's Theorem)**.** *Let $P$ be a simple polygon in the plane, each of whose vertices is a lattice point. (That is, for each vertex $V$ on the boundary of $P$, both the $x$- and $y$-coordinates of $V$ are integers.) Let $I$ denote the number of lattice points in the interior of $P$, and let $B$ denote the number of lattice points on the boundary. (I.e., $B$ counts the number of lattice points that lie on the edges of the boundary of $P$.) Then the area $A$ of $P$ is given by the formula*

$$A = I + \frac{B}{2} - 1.$$

With these preliminaries out of the way, we proceed to the main result from this supplement:

**Theorem 6.8.** *Let $\alpha$ be a real number, and let $\frac{P_k}{Q_k}$ be a convergent to $\alpha$. If $\frac{P}{Q}$ is any rational number with $P, Q \in \mathbf{Z}$, $Q > 0$, and $Q < Q_k$, then*

$$\left|\alpha - \frac{P_k}{Q_k}\right| < \left|\alpha - \frac{P}{Q}\right|.$$

*That is, we cannot find a better rational approximation to $\alpha$ than $\frac{P_k}{Q_k}$ using a smaller denominator.*

*Remark.* Theorem 6.8 appears as Exercise #4.3(b) on the aforementioned worksheet from October 16, 2021.