Topology of Surfaces

Most of this material is from *The Shape of Space* by Jeff Weeks. The word searches were taken from Torus Games at http://geometrygames.org/TorusGames/index.html . You can download the Torus Games app for many other games.

1 Geometry vs. Topology

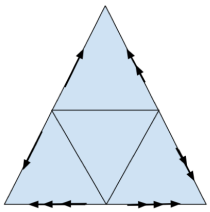
- The properties of an object that change when you bend or stretch it are the geometry of the object.
  - For example, angles and distances are parts of geometry but not topology.
- The properties of an object that stay the same when you bend or stretch it are called the topology of the object.
  - Two objects are considered the same topologically if you can deform one into the other without tearing, cutting, fusing, or other violent actions.

"Mug and Torus morph". Licensed under Public Domain via Wikimedia Commons

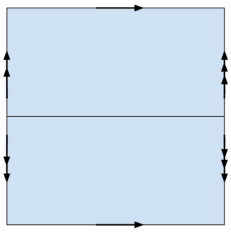
1. Whic surfaces are topologically the same? Match a surface on the top row with a surface on the bottom row.

2 Gluing Diagrams

2. What topological surface do you get when you glue (or tape) the edges of the triangle together as shown?

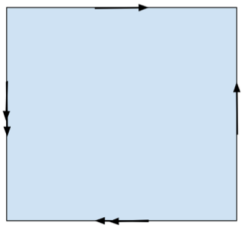


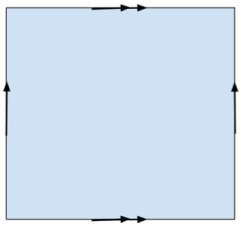
3. What do you get when you glue the edges of the square together like this?



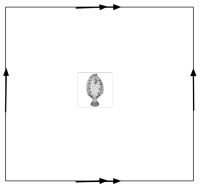
Don't glue the interior parts of the square together, just the edges!

4. What surface is this? And this?





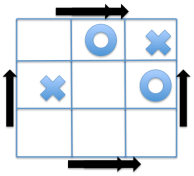
5. What happens as this 2-dimensional creature travels through its tiny universe?



What does it see when it looks forwards? Backwards? Left? Right? (See Torus Games animation.)

3 Tic-Tac-Toe on a Torus

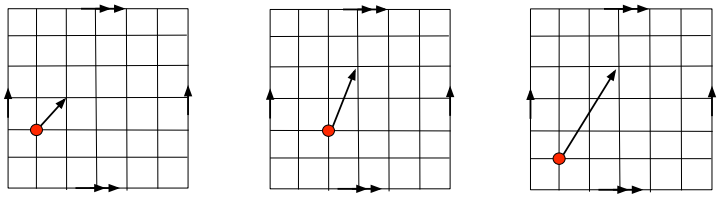
6. Where should X go to win? What if it's O's turn?



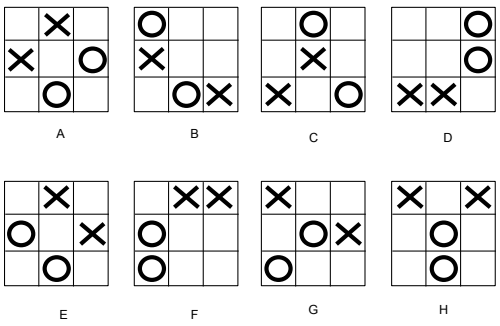
7. Play some games of Tic-Tac-Toe on the Torus with a classmate.

4 Torus Problems

8. A ladybug on a torus walks in a straight line until she returns to her starting point. Draw her path and find the length of her path if:
   (a) she walks 1 unit northward for every 1 unit eastward, as shown at the left.
   (b) she walks 2 units northward for every 1 unit eastward as shown in the center.
   (c) she walks 3 units northward for every 2 units eastward as shown at right.



9. Which of the following positions are equivalent in torus tic-tac-toe?

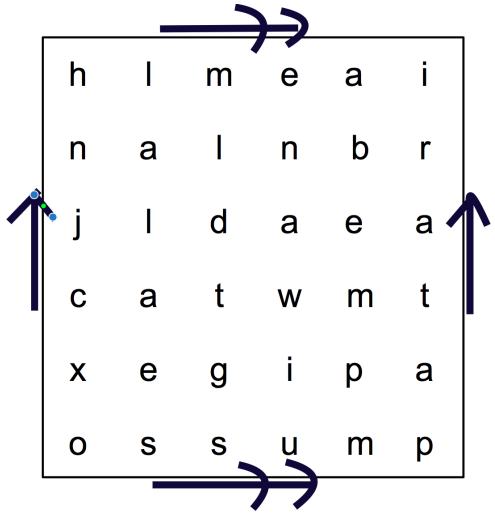


10. How many essentially different first moves are there in torus tic-tac-toe?

11. Is there a winning strategy for the first player in Tic Tac Toe on the torus? That is, is it possible for the first player to win no matter what the second player does?

12. A Cat's Game in Tic Tac Toe is a game where neither side wins, even though the board is filled up with X's and O's. Is it possible to have a Cat's Game in Tic Tac Toe on the torus?

13. Try this word search on the torus. Remember that the top and the bottom are glued together and the left and right sides are glued together.

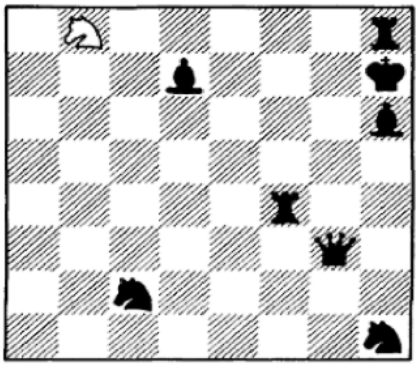


bat
camel
cat
jaguar
llama
ox
panda
pig
possum
puma
rat
seal
shrew
tapir

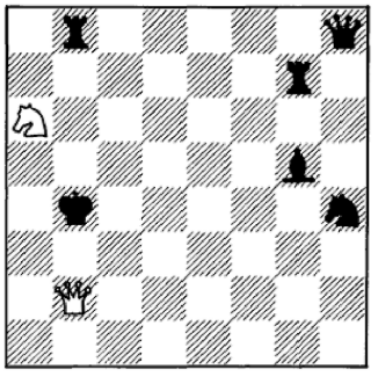
4.1 Extra Problems

14. A society of Flatlanders lives on a torus that can be described by gluing a rectangle is 10 km long and 8 km wide. For maximal effectiveness, the fire stations should be placed as far apart as possible. Where might the Flatlanders put them?

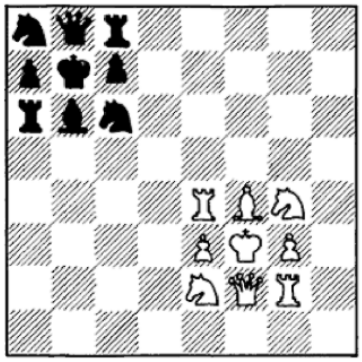
15. Consider the position in chess on a torus shown below. Which black pieces does the white knight threaten? Which black pieces threaten it?



16. Which black pieces are threatened by *both* the white knight and the white queen?



17. Play a few games of torus chess with a classmate. The usual starting position just won't do for torus chess (try it and you'll see why). Instead, either use the starting position below or make up a starting position of your own. All the pieces move normally except the pawns: a pawn moves one space forward, backward, to the left or to the right, and captures by moving one space on any diagonal.

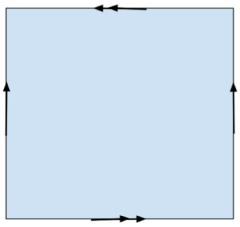


18. When a bishop goes out the upper right hand corner of a torus chessboard, where does he return?

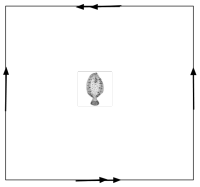
19. In torus chess, can a knight and a bishop simultaneously threaten each other?

5 The Klein Bottle

20. What surface do you get when you glue together the sides of the square as shown?



21. What happens as this creature travels through its Klein bottle universe?

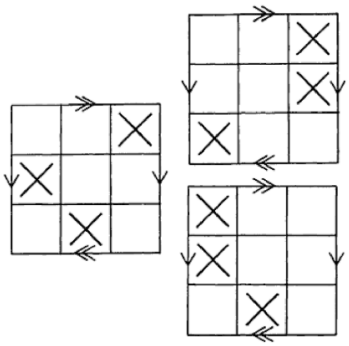


- A path that brings a traveler back to his starting point mirror-reversed is called an orientation-reversing path.
- A surface that contains an orientation-reversing path is called non-orientable.

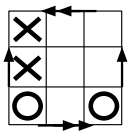
22. Can you find more than one orientation reversing path in this surface?

6 Tic-Tac-Toe on a Klein Bottle

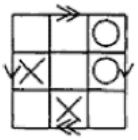
23. Which of these are winning positions in Klein bottle Tic-Tac-Toe?



24. Where can X go to win immediately in Klein bottle Tic Tac Toe?

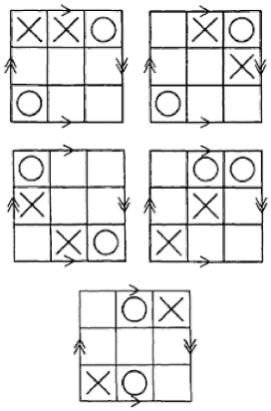


25. Where should X go to win?

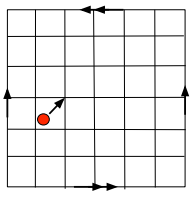


26. Play a few rounds of Klein bottle Tic-Tac-Toe with a classmate.
    - Is there a winning strategy?
    - Is it possible to get a cat's game?
    - How many essentially different first moves are there?

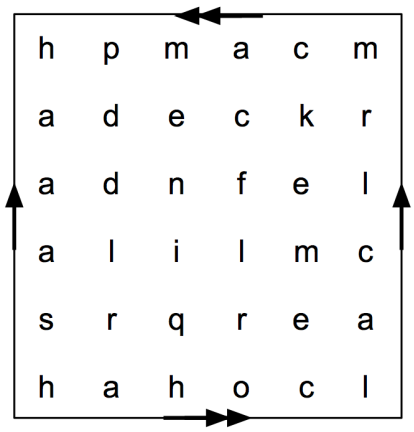
27. What are the best moves for X in these positions?



28. A ladybug on a Klein bottle walks in a straight line until she returns to her starting point. She walks 1 unit northward for every 1 unit eastward. Draw her path.



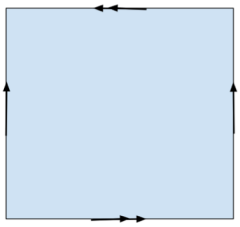
29. Try this word search on the Klein bottle. The arrows show how the sides are glued together.



ash
birch
cedar
elm
fir
larch

lilac
maple
oak
palm
pine
poplar

30. What happens when you cut a Klein bottle in half? Hint: It depends on how you cut it.



Cutting a Klein bottle in half - animation

Cutting a Klein bottle in half - IRL

7 3-manifolds

31. Is there a 3-dimensional analog to the torus? What about the Klein bottle?