Irrational Numbers

Abstract

In this session, we shall explore *irrational numbers*. We begin with some basic properties of the *rational numbers*; for example, the sum and product of two rational numbers is also a rational number. Beginning with basic properties of rational and irrational numbers, we next move to multiple different proofs that $\sqrt{2}$ is irrational. We then generalize these ideas to prove large classes of real numbers are irrational, including many roots, logarithms, and some specific trigonometric values.

0 Warmup Exercises

0.1 What is a *rational number*? What is an *irrational number*? Can you give examples of each type of number? (We'll be *proving* particular numbers are irrational beginning in Section 2, so don't feel that you need to prove something is irrational at this stage.)

0.2 Let q be a rational number. What is the *reduced form* or *lowest terms* form of a fraction? What is the reduced form for the fraction $q := -\frac{12}{30}$?

0.3 What is *proof by contradiction*

Bonus: Can you provide an example of a true mathematical statement that you can prove using proof by contradiction?

1 Introduction: Rational and Irrational Numbers

Let us begin with some very basic properties of the rational and irrational numbers.

1.1 Let q be a rational numbers.[1] Prove that $-q$ is a rational number. Further, prove that if q is rational and $q \neq 0$, then $1/q$ is rational.

Note: You may assume that if a is an integer, then $-a$ is also an integer. Further, you may assume that a rational number of the form a/b is zero if and only if $a = 0$.

1.2 Let q, r be rational numbers. Prove that $q + r$ and qr are also rational numbers. (That is, the set of rational numbers is *closed* under both addition and multiplication.) Combining this with the result of Exercise #1.1, it follows that for all rational q, r, $q - r$ is rational. Further, if q, r are rational and $r \neq 0$, then q/r is also rational.

Note: As in the Note to Exercise #1.1, you may assume that the sum and product of two integers is also an integer. Furthermore, you may assume that if m, n are *nonzero* integers, then mn is also a *nonzero* integer. Further, a rational number of the form a/b is zero if and only if $a = 0$.

1.3 Assume that q is a rational number and x is an irrational number. Is $q + r$ rational or irrational? Is qr rational or irrational? Explain.

Note: It will be natural to used proof by contradiction, the technique mentioned in Exercise #0.3.

[1] *Notation:* $\mathbb{Q}$ denotes the set of all rational numbers, so this is equivalent to writing "Let $q \in \mathbb{Q}$. Prove that $-q \in \mathbb{Q}$. Further, if $q \in \mathbb{Q}$ and $q \neq 0$, prove that $1/q \in \mathbb{Q}$."

1.4 (a) Is there an irrational number x such that $-x$ is a rational number?

(b) Is there an irrational number x such that $1/x$ is a rational number?

(c) Are there irrational x, y such that $x + y$ is rational?
Note: For this exercise, you may assume that $\sqrt{2}$ is irrational.

(d) Are there irrational x, y such that xy is rational?
Note: For this exercise, you may again assume that $\sqrt{2}$ is irrational.

2 Proving $\sqrt{2}$ Is Irrational

Now that we have some basic understanding of properties of rational and irrational numbers, let us finally establish that some irrational numbers exist! In this section, we shall provide multiple proofs that $\sqrt{2}$ is an irrational number.

2.1 The following proof by contradiction has been attributed to Hippasus, an ancient Greek philosopher. We proceed with a proof by contradiction.

Assume, instead, that $\sqrt{2}$ is a rational number. That is, we assume $\sqrt{2} = a/b$, where a, b are integers with $b \neq 0$. By Exercise #0.2, we may assume without loss of generality that a/b is a reduced fraction, meaning that a and b have no common factor. Squaring both sides of $\sqrt{2} = a/b$, we obtain $2 = a^2/b^2$, and thus

$$a^2 = 2b^2. \tag{2.1}$$

(a) Prove that if a, b are as above, then a must be even. That is, there exists some integer a' such that $a = 2a'$.

(b) Substituting $a = 2a'$ into (2.1), prove that b must also be even. That is, there exists some integer b' such that $b = 2b'$.

(c) If a, b are integers such that (2.1) holds, then $\gcd(a, b) \neq 1$.

(d) Conclude that $\sqrt{2}$ cannot be a rational number.

2.2 Can you modify Hippasus' argument in Exercise #2.1 to prove that $\sqrt{3}$ is also irrational? What about $\sqrt{5}$? Where does Hippasus' argument break down in trying to adapt it to determine whether $\sqrt{4} = 2$ is irrational?

2.3 Next, we prove that $\sqrt{2}$ is irrational using prime factorizations and *The Fundamental Theorem of Arithmetic*: every positive integer greater than 1 can be written as a product of prime numbers, and that product is unique up to ordering.

Assume, again, that $\sqrt{2}$ is rational, with $\sqrt{2} = a/b$, where a/b is the reduced fraction for $\sqrt{2}$. As in Exercise #2.1, we conclude (2.1): $a^2 = 2b^2$.

(a) Considering the left-hand side of (2.1), show that the *multiplicity* of 2 as a divisor of a^2 must be even.

(b) Considering the right-hand side of (2.1), show that the multiplicity of 2 as a divisor of $2b^2$ must be odd.

(c) Conclude from Exercises #2.3(a)–2.3(b) that $\sqrt{2}$ is irrational.

2.4 Our next proof that $\sqrt{2}$ is irrational shall be geometric. Assume that $\sqrt{2} = a/b$, where the right-hand side is such that a/b is the reduced fraction for $\sqrt{2}$. We once again note

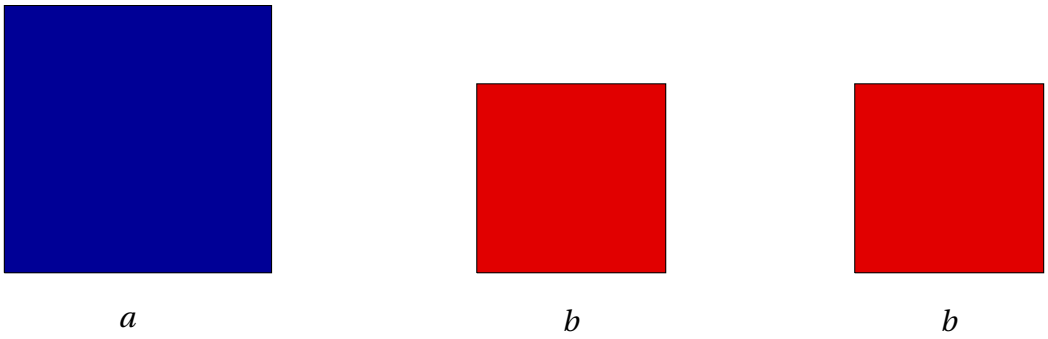

Figure 2.1: Minimal squares of integer sides lengths a, b such that $a^2 = b^2 + b^2$.

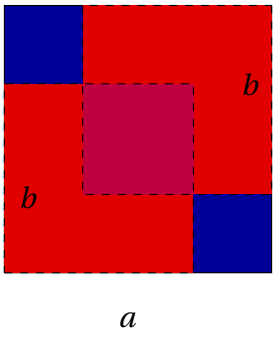

Figure 2.2: Red squares from Figure 2.1 placed atop the blue square

that (2.1) holds: $a^2 = 2b^2$. Form one square with sides of length a, and two with sides of length b, as in Figure 2.1

Next, rearrange the red squares to lie, with overlap, inside the blue square from in Figure 2.1. Here, the central purple square is the overlap of the two red squares inside the larger blue square.

(a) Consider the overall area of the blue square with side length a, itself covered by the two red squares having side length b. Prove that the area of the central purple overlap is equal to the sum of the blue uncovered squares above.

(b) What are the side lengths of the central purple square and each blue corner square?

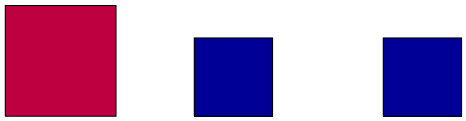

Figure 2.3: New squares of integer side length, similar to those in Figure 2.1.

(c) Show that the conclusion from Exercise #2.4(b) provides a *smaller* set of squares with integer side lengths as above:

2.5 Finally, we sketch a proof that $\sqrt{2}$ is irrational using the theory of *continued fractions*.

(a) Prove that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}}}. \tag{2.2}$$

(b) If q is a positive rational number, then q has a *finite* continued fraction expansion of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}},$$

where a_0 is an integer, and $a_1, a_2, \ldots, a_n$ are all positive integers.

(c) Assume that the continued fraction expansion of any positive real number is unique—at least up to whether it has finite length.[2] Conclude from Exercises #2.5(a)–2.5(b) that $\sqrt{2}$ must be irrational.

2.6 Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

3 The Rational Root Theorem

The result from Exercise #2.6 suggests that we can verify that a very large collection of numbers of the form $\sqrt{m}+\sqrt{n}$, where m,n are non-square positive integers, are all irrational. In this section, we explore a theorem that simplifies our ability to establish that many real numbers are irrational.

Theorem 3.1 (The Rational Root Theorem)**.** *Let n be a positive integer, and let $a_n, a_{n-1}, \ldots, a_1, a_0$ be integers with $a_n \neq 0$. Consider the polynomial*

$$p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0. \tag{3.1}$$

Let q be a rational number $q = a/b$ such that a, b are integers, $b \neq 0$, and $\gcd(a,b) = 1$.

If q is a root or zero of $p(x)$—i.e., if $p(q) = 0$, meaning we get 0 when we "plug q into $p(x)$"—then

- *a, the numerator of q, is a divisor[3] of the constant term a_0, and*
- *b, the denominator of q, is a divisor of the leading term a_n.*

[2] Strictly speaking, the continued fraction representation isn't *quite* unique. (For example, we have $3 = 2+1/1$.) But other than such trivial exceptions, we can indeed select a unique "standard" continued fraction representation for every rational number. Further, it is true that the positive real number x has a continued fraction expansion of finite length if and only if x is a rational number.

[3] Note that we are simply requiring that $\gcd(a,b) = 1$, *not* that a/b is lowest terms in the sense of Exercise #0.2. In particular, we may have either of a and b is negative, not our lowest-terms convention that b must be strictly positive.

That is, the only rational numbers a/b (with $\gcd(a,b) = 1$) that are even candidates *to be roots of $p(x)$ must be such that the numerator is a divisor of the constant term, and the denominator is a divisor of the leading coefficient.*

Example 3.2. Consider the polynomial

$$p(x) := 7x^3 - x^2 - 14x + 2. \tag{3.2}$$

Then in the language of Theorem 3.1, we have

- $n := 3$

  I.e., the polynomial *degree* of $p(x)$ is 3 because the highest nonzero term is $7x^3$.

- $a_0 := 2$

  I.e., the constant term is 2.

- $a_1 := -14$

  I.e., the coefficient of x is -14.

- $a_2 := -1$

  I.e., the coefficient of x^2 is -1.

- $a_3 := 7$

  I.e., the leading coefficient, which is the coefficient of x^3, is 7.

Then by the Rational Root Theorem, the *candidates* for rational roots to $p(x)$ are precisely

$$\frac{\pm 1}{\pm 1}, \frac{\pm 2}{\pm 1}, \frac{\pm 1}{\pm 7}, \text{ and } \frac{\pm 2}{\pm 7}.$$

By plugging in each of these eight values, we see that $p(1/7) = 0$, and none of the other seven candidates is an actual root. In fact, we have

$$p(x) = (7x-1)(x+\sqrt{2})(x-\sqrt{2}),$$

so $p(x)$ has the three roots $1/7$, $-\sqrt{2}$, and $\sqrt{2}$. (Note, in particular, that $p(x)$ does have irrational roots not among the rational candidates provided by the theorem.)

3.1 Find a polynomial $p(x)$ with integer coefficients such that $\sqrt{2}$ is a root. Use the Rational Root Theorem to conclude that $\sqrt{2}$ is irrational.

3.2 Mimicing the strategy in Exercise #3.1, prove that $\sqrt{2}+\sqrt{3}$ is irrational.

3.3 We now use the Rational Root Theorem to show that a particular trigonometric value, $\cos 20^\circ = \cos \pi/9$ (in radians), is irrational. (This question will assume results from trigonometry; just skip it if you haven't yet studied any trig.) This exercise is taken from Chapter 5 of [2].

(a) What is $\cos 60^\circ = \cos \pi/6$ (in radians)?

(b) Using the identity

$$cos3\theta = 4(\cos\theta)^3 - 3\cos\theta$$

for all θ, conclude that $\cos 20^\circ$ is a root of the polynomial

$$p(x) = 8x^3 - 6x - 1.$$

(c) Using the Rational Root Theorem, prove that $\cos 20^\circ$ is irrational.

4 Miscellaneous Irrational Numbers

We conclude with a few final examples of irrational numbers. This section will consider logarithms and the constant e; if you haven't yet seen that, just skip this section.

4.1 Recall that if x, y are positive numbers, with $y \neq 1$, then

$$\log_y x = z \text{ if and only if } y^z = x. \tag{4.1}$$

That is, logarithms and exponentials are inverse operations.

Example 4.1. We have

$$\log_2 8 = 3 \text{ because } 2^3 = 8.$$

Using the Fundamental Theorem of Arithmetic, prove that

$$\log_{10} 3$$

is irrational.

4.2 Define e to be the constant

$$e := \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \cdots \tag{4.2}$$

Then e is irrational.

References

[1] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, Walton Street, Oxford OX2 6DP, fifth edition, 1979.

[2] Ivan Niven. *Numbers: Rational and Irrational*. New Mathematical Library. Random House and The L. W. Singer Company, New York, 1961.

[3] Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery. *An Introduction to the Theory of Numbers*. John Wiley & Sons, Inc., New York, fifth edition, 1991.