Constructible Points and Constructible Numbers, Part 1

Abstract

In this session, we shall introduce mathematical constructions by straightedge and compass, thereby defining *constructible points* and *constructible numbers*. With such techniques, we can perform classical constructions, such as finding the midpoints of a given segment, constructing certain perpendicular or parallel lines, and bisecting angles. Over the next two sessions, our goal is to explore which points and numbers are—and are not—constructible, and how we can determine constructibility.

Prior experience with geometry, such as similar triangles, will be beneficial, but this session is intended to be largely self-contained.

0 Warmup

Exercises:

0.1 If $\triangle ABC$ and $\triangle DEF$ are triangles, what does it means that $\triangle ABC$ and $\triangle DEF$ are *congruent triangles*? What methods are available to prove two triangles are congruent? Specifically: what axioms/postulates or theorems are available as tools to prove that $\triangle ABC \cong \triangle DEF$?

0.2 If $\triangle ABC$ and $\triangle DEF$ are triangles, what does it means that $\triangle ABC$ and $\triangle DEF$ are *similar triangles*? Specifically: what axioms/postulates or theorems are available as tools to prove that $\triangle ABC \sim \triangle DEF$?

0.3 Let $\triangle ABC$ be a triangle. What is a *median* in $\triangle ABC$? What is an *angle bisector* of $\triangle ABC$? What is an *altitude* of $\triangle ABC$?

0.4 What does it mean for a circle $\odot O$ to be tangent to a line ℓ?

1 Introduction: Straightedge and Compass Constructions

You will be provided with a compass and a straightedge.[1] In considering constructions of points in the plane and numbers, we may make the following constructions:

Definition 1.1. In the plane, we are given two distinct, specified points P_0, P_1, and we choose coordinates such that $P_0 := (0,0)$, $P_1 := (1,0)$. (In particular, the length P_0P_1 of the segment $\overline{P_0P_1}$ is 1.) By definition, P_0 and P_1 are constructible points.

Assume that P, Q are already-constructed points. Then using our straightedge and compass, we may

(a) use the straightedge to draw the line $\overleftrightarrow{PQ}$.[2]

(b) use the compass to draw a circle $\odot P$ with center P and radius $\overline{PQ}$; and

(c) for any already-constructed line or circle that intersects another already-constructed line or circle, we may specify any point of intersection R.

We say that any point R distinguished as in Definition 1.1(c) is a *constructible point*. Further, we define a real number x to be a *constructible number* if and only if we can construct two points P and Q such that that $|x| = PQ$. That is, constructible numbers are precisely those arising as distances between constructible points, up to sign.

That is, R is a constructible point if and only if we locate R after finitely many constructions of the form given in Definition 1.1(a)–1.1(c), beginning with our given points P_0 and P_1.

Remark. I should emphasize a distinction between a *constructible line* or *constructible circle* and a *constructible point*.

Given a constructed point O and constructible number r, we can construct the circle C with center O and of radius r. However, though the circle as a whole is constructible, only

[1] In the classical formulation of constructions with straightedge and compass, the straightedge is *unmarked*, unlike the rulers or protractors provided today.

[2] For our purposes, we may imagine that our straightedge is infinitely long, so that we may construct the *line* $\overleftrightarrow{PQ}$ rather than simply the *line segment* $\overline{PQ}$;

those points that arise as the intersection of a constructed line or circle with another constructed line or circle are constructible *points*. This technical distinction may be confusing: we may be able to construct a like ℓ or a circle C, and those figures are constructible in that sense. Paradoxically, though, not every point on ℓ or C is necessarily a constructible point!

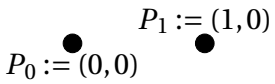
The basic distinction is that a constructible point isn't simply something that lies on a constructible line or point. Rather, *to be a constructible point*, it must be the intersection of a constructible line or point with another constructible line or point.

One of the classical problems from antiquity, going back at least to Euclid's *Elements*, is to determine precisely which points in the plane are constructible points, and precisely which real numbers are constructible numbers.

Let us begin with some introductory exercises:

Exercises:

1.1 Given our initial points $P_0 := (0,0)$ and $P_1 := (1,0)$ in the plane, explain how to construct the point $(2,0)$. Can you construct $(3,0)$? What about $(-1,0)$? Explain why we can therefore construct every point $(n,0)$ on the x-axis, where n is any integer.



1.2 Assume that P, Q are distinct, constructible points as given. Show how to construct a point R such that $\triangle PQR$ is equilateral. Is R unique?

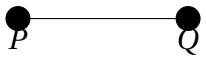
1.3 Let P, Q be distinct constructible points. Construct a line ℓ through P which makes an angle 60° with $\overleftrightarrow{PQ}$.

1.4 Let x, y be constructible numbers. Prove that $x \pm y$ is a constructible number. More generally, prove that for all integers m, n, the number $mx + ny$ is a constructible number.

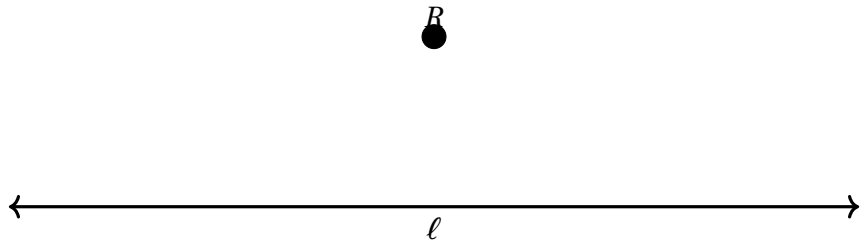
2 Basic Constructions

In this section, we hope to develop some basic constructions that will be useful in general settings. In particular, we would like to find the perpendicular bisector of a given line segment, produce a line through a given point that's perpendicular to a given line, and produce a line through a given point that's parallel to a given line. Using these building blocks, we shall show that all the integer lattice points of the form (m,n) are constructible points, thereby enabling many additional constructions.
Exercises:

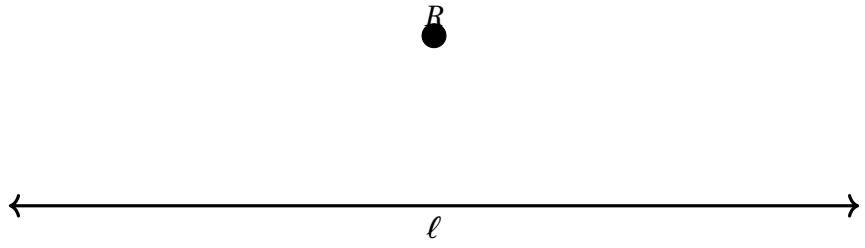
2.1 Given distinct constructible points P,Q, explain how to construct the perpendicular bisector to $\overline{PQ}$. In particular, explain how this shows how to construct the midpoint M of $\overline{PQ}$.



2.2 Let ℓ be a given line. If R is a given constructible point, where R does not lie on ℓ, construct the line $\ell_\perp$ through R such that $\ell_\perp$ is perpendicular to ℓ.



2.3 Let ℓ be a given line. If P is a given constructible point, where P does not lie on ℓ, construct the line $\ell_{\parallel}$ through P such that $\ell_{\parallel}$ is parallel to ℓ.



2.4 Show that every point of the form (m,n), where m and n are both integers, is a constructible point. (Recall: we are given that $(0,0)$ and $(1,0)$ are constructible by definition.)

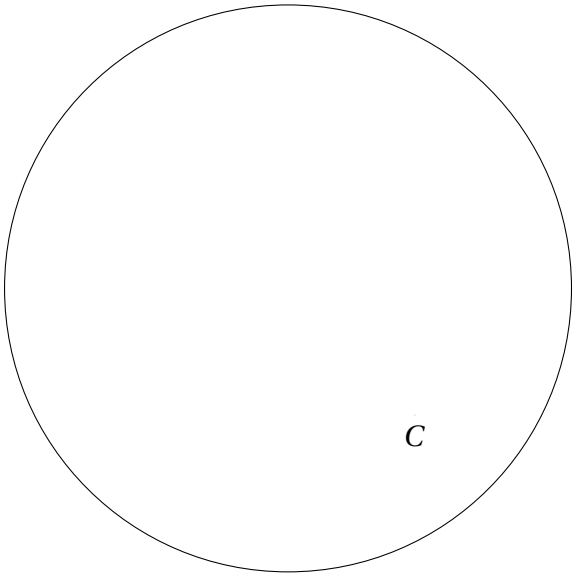
2.5 Prove that $\sqrt{2}$ is a constructible number. Can you also show that $\sqrt{3}$ is constructible?

Hint: Can you show that $\sqrt{2}$ is the length of a line segment whose endpoints have integer coordinates? A different approach, though one familiar from Section 1 may be needed for showing $\sqrt{3}$ is also constructible. (Or, alternatively, use the constructibility of $\sqrt{2}$, 1, and the result of Exercise #3.6.)

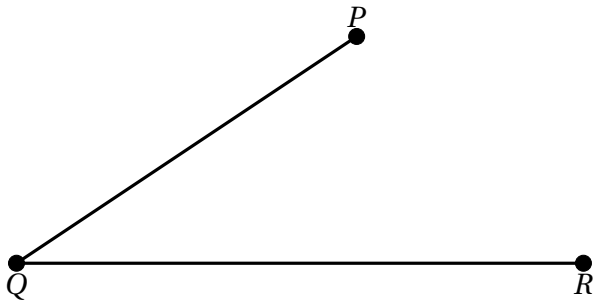
2.6 Prove that we can construct an angle of measure 45°.

Hint: Can you realize this as the angle of a triangle whose vertices are all constructible points?

2.7 Given a circle C, construct its center O.



2.8 Let P, Q, R be given constructible points. Explain how to construct the angle bisector ℓ of $\angle PQR$. (You may want to consider separately the case where $m\angle PQR = 180°$.)



3 Intermediate Constructions

Much of our constructions from Sections 1–2 used methods of congruent triangle. In this section, we continue using that, and we also develop techniques using similar triangles. This will also use certain results about circles and right triangles, many of which themselves can be proven using similar triangles.
Exercises:

3.1 Let a, b constructible numbers, Prove that ab is a constructible number.

3.2 Let $x > 0$ be a given constructible number, so that the point $(x, 0)$ is a constructible point. Prove that the point $\left(\frac{1}{x}, 0\right)$ is constructible point, and therefore $\frac{1}{x}$ is a constructible number.

3.3 Let a, b constructible numbers with $b \neq 0$. Prove that $\frac{a}{b}$ are both constructible numbers.

3.4 Prove that for all rational numbers[3] r, s, the points (r, s) is constructible in the plane.

[3] By definition, r is a *rational number* if and only if there exist integers a, b, $b \neq 0$, such that $r = a/b$. That is, a number is rational if and only if it is the quotient of two integers, and the denominator is nonzero.

3.5 Prove that if a is a positive constructible number, then $\sqrt{a}$ is also a constructible number. More generally, prove that if a, b are positive constructible numbers, then $\sqrt{ab}$ is a constructible number.

Hint: Consider a semicircle of diameter $1+a$. Can you use this to form similar triangles for which the length of some relevant segment is $\sqrt{a}$?

3.6 Let a, b be constructible numbers. Prove that $\sqrt{a^2+b^2}$ is a constructible number.

4 Preview of Our Next Session

Thus far, we have focused on which points and numbers *are* constructible. That suggests a natural question: are there any points or numbers that are *not* constructible? If so, can we provide some examples? Better still, can we provide a complete characterization of which points/numbers are and are not constructible?

This will be the focus of our next session (on December 2, 2023). In particular, we'll show that $\sqrt[3]{2}$ is an example of a length that is not constructible. There are additional impossibility theorems we may explore, though full, rigorous detail will be beyond the scope of our time availble. For example, in general, it is impossible to trisect an arbitrary angle using (unmarked) straightedge and compass alone. It is also impossible to construct a square S whose area is equal to that of a circle of given radius; this is the so-called "squaring the circle" made famous by mathematicians from ancient Greece.

For additional exercises, I recommend exploring Euclidea (https://www.euclidea.xyz), availble both as a free website and as an app. There, you can explore other constructions with straightedge and compass. For example: given a rhombus, can you construct an inscribed circle that is tangent to each of its four sides? Given three distinct points, can you construct a fourth so that all four are the vertices of a parallelogram? Euclidea not only asks

you to find *some* solution, but it also challenges you to find optimally *efficient* ones, requiring as few moves as possible.

Thanks for coming, and we hope to see you again in two weeks!