**Chapel Hill Math Circle**
**Session 8 – January 20, 2024: Breaking Numbers**
**Beginners' Group (grades 1-3), 10:30-11:30a**
**Mr. Barman** – dilip@trianglemathinstitute.com

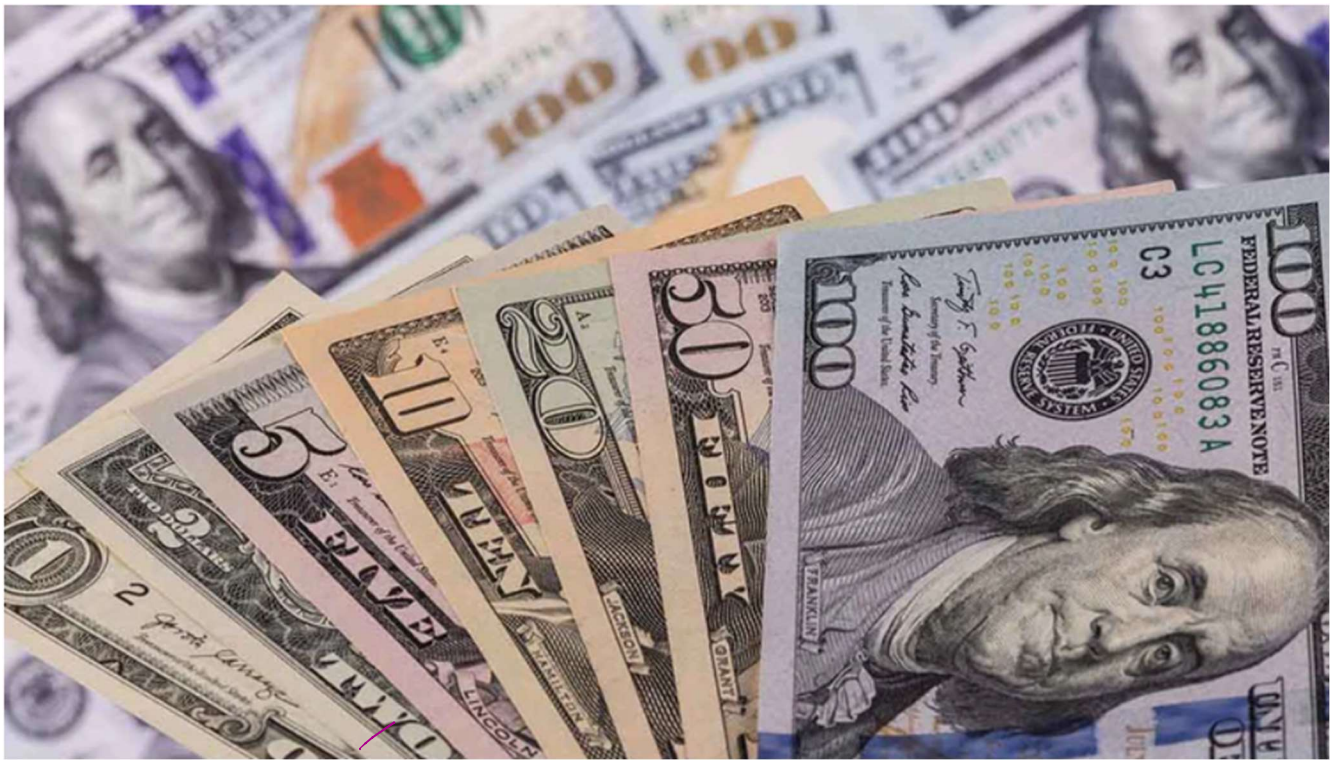
*Supplies needed: colored rectangles that need to be cut out and are included here.*

Happy 2024! I hope that you had a good holiday season and are having a good new year so far. Welcome (back) to Chapel Hill Math Circle! We're glad that you are here to have some fun!

**Let's look at making and breaking numbers – starting with breaking money**

What makes up numbers? More numbers. Let's break down 10. One way is 5 + 3 + 1 + 1. There are many more ways. Does this remind you of making change? Let's discuss "breaking money" first.

What does it mean to make change? Money comes in different "denominations" like $1, $2, $5, $10, $20, $50, and $100 as shown in the image below[1]. Do you know what it means to "break" money? Why do people do that?




[1] from ribaostore.com/blogs/news/how-much-does-a-dollar-bill-weigh, accessed Jan. 16, 2024

**Pieces of eight**

We don't cut our money; in fact, it is illegal. But at one time people were allowed and even encouraged to do that. If you look at the picture below[2], you will see the silver Spanish dollar used in the 16th century. It was also called "pieces of eight" because you cut use a metal cutter to cut it in up to 8 pieces!

[2] from seahistory.org/sea-history-for-kids/pieces-of-eight, accessed Jan. 16, 2024

**Breaking money?**

As far as I know, nobody cuts money nowadays. But say that you have a $20 bill[3]. Why might you want to give it to a bank and ask to have it "broken" into, maybe:

- 20 $1 bills
- 4 $5 bills
- 2 $10 bills?

Or maybe you want some combination like:

- 1 $10 bill, 1 $5 bill, and 5 $1 bills
- 1 $10 bill and $10 in quarters (how many quarters would this be?)
- 1 $5 bill and 15 $1 bills

Can you think of why somebody might do this?

[3] This Harriet Tubman $20 bill artist's concept is from washingtonpost.com/politics/harriet-tubman-20-bill/2021/06/03/62443b5c-bcd1-11eb-9c90-731aff7d9a0d_story.html, accessed Jan. 16, 2024.

Sometimes people break money because they need change for a vending machine. Or maybe they are going on a trip and want to give a luggage helper (porter) a “tip” of $5 but the cash that they got from an ATM machine was in $20 bills.

I don’t give our candy at Halloween but I give out quarters. 4 quarters make a dollar so 40 quarters make $10 and that’s how banks package quarters. I typically go to a bank a little before the holiday and give them maybe $40. They take the $40 and return to me 4 wrappers full of quarters[4] ($10 + $10 + $10 + $10 = the $40 I’ve given them). These 4 rolls total 40+40+40+40 = 80+80 = 100+60 = 160 quarters. So I’d be ready for up to 160 trick or treaters. What I’ve done is “broken” my $40 (maybe 2 $20 bills) into 160 quarters.

[4] This image of a roll of quarters is from alibaba.com/product-detail/Custom-design-assortment-pack-coin-roll_1600411337546.html, accessed Jan. 16, 2024.

**Let's break numbers**

Let's get back to breaking down 10. One way is 5 + 3 + 1 + 1, as we mentioned before. How about these ways?

- 1+1+1+1+ … + 1 (10 1s)
- 3 + 1 + 2 + 1 + 2 + 1
- 4 + 2 + 3 + 1

There are even more ways. What do you think of the list so far? How about my approach?

**Let's improve our approach**

Do you agree that I should be more organized? Is this a better start?

- 10 1s
- 8 1s and a 2
- 7 1s and a 3
- 6 1s and a 4
- 6 1s and 2 2s

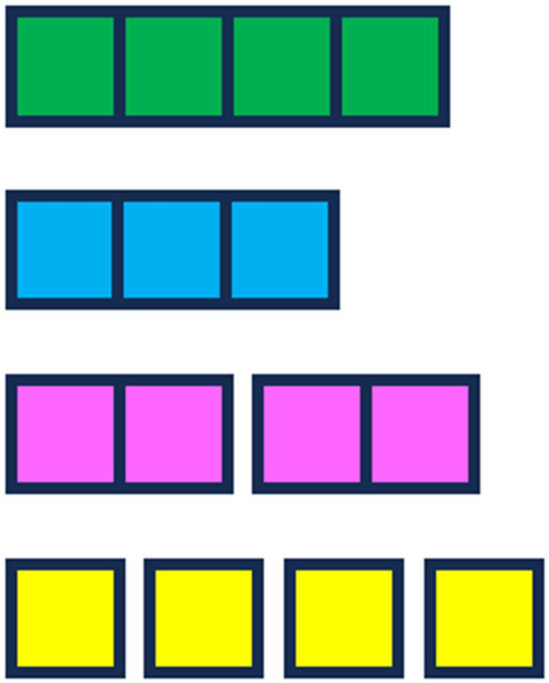
But what about this?

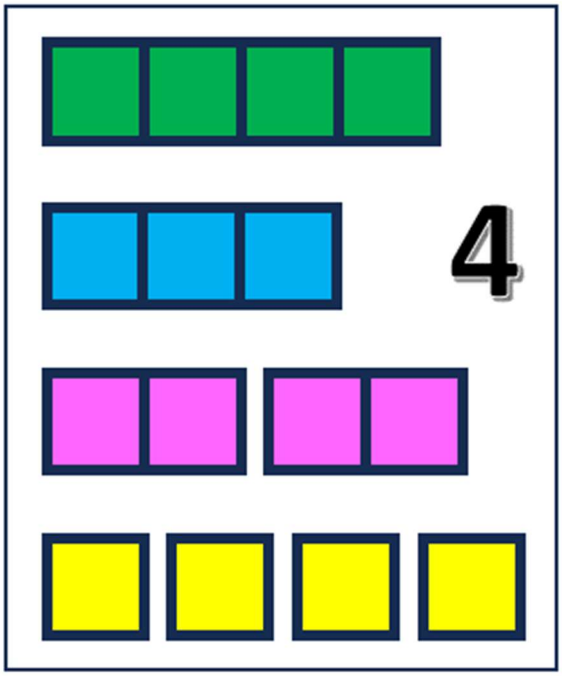
- 12 1s – 2
- 8 + 3 – 1
- 5 + 3 + 4 – 2

What's wrong with this approach? The problem is that if we include subtraction there are an infinite number of answers. And the problem with breaking down 10 is that it is a big initial problem.

**You should have some sets of rectangles already cut out**

They are also on the next page. We will use sets labeled 4, 5, 6, and 7. Starting with the set labeled 4, you should cut out and have something like these 1 + 1 + 2 + 4 = 8 rectangles.

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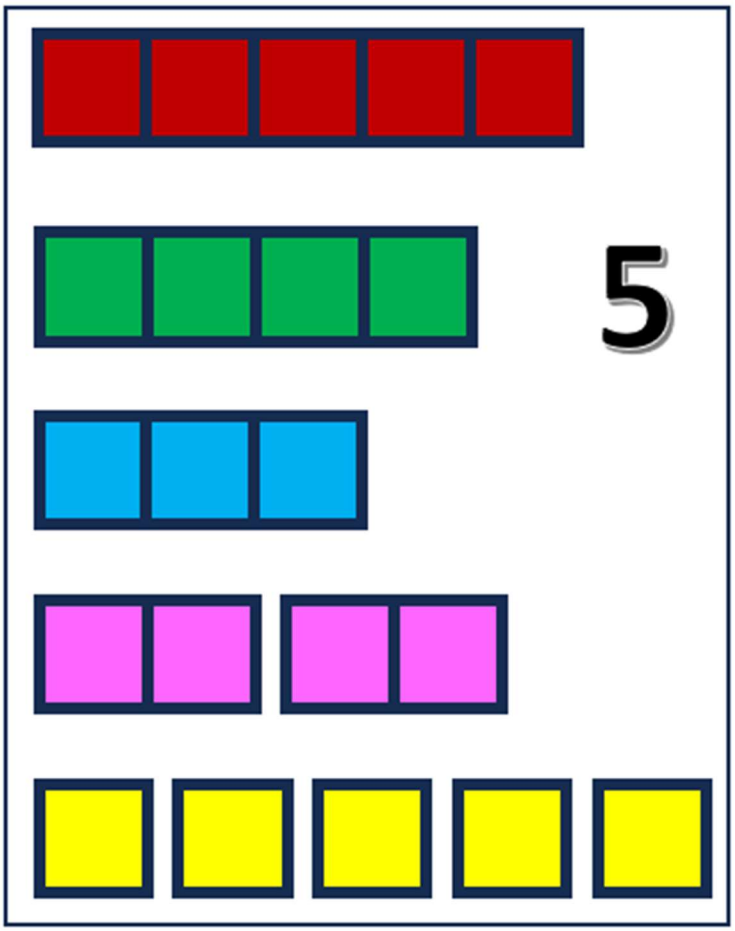
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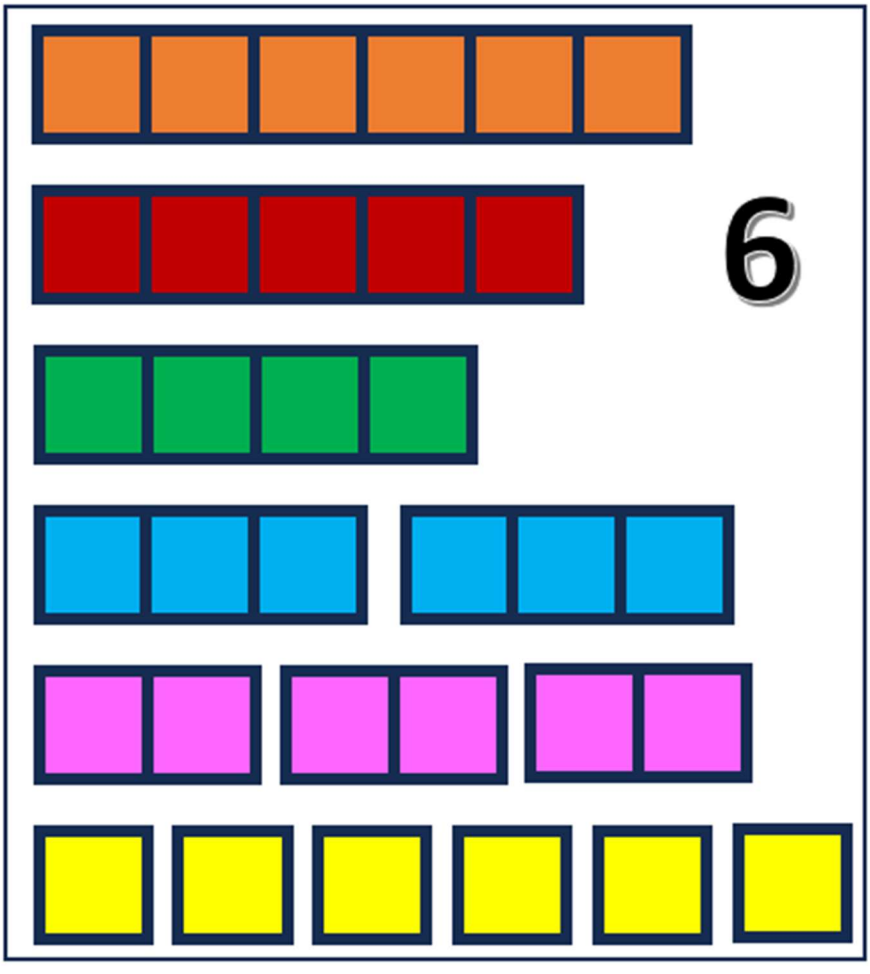
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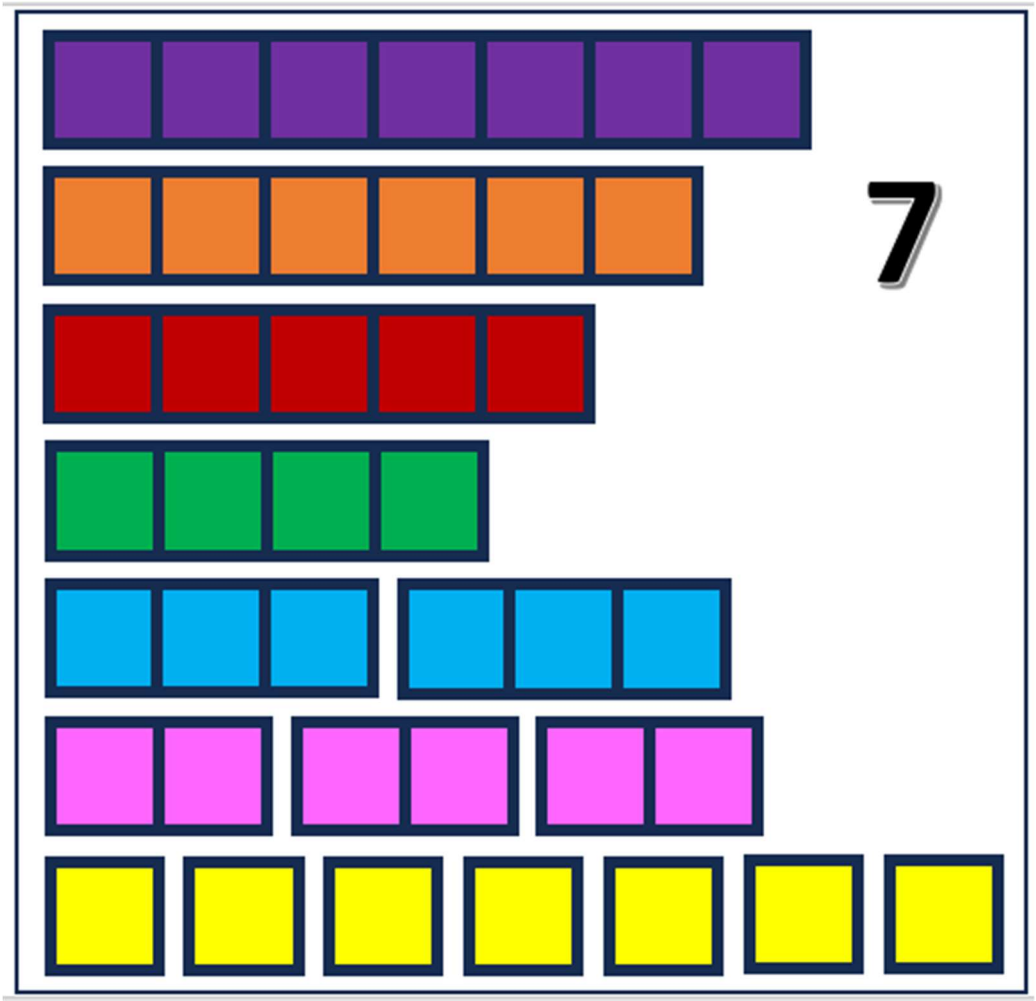
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**Now let's play with squares connected into rectangles**

Let's look at ways to break up 4. Start with these 8 rectangles; one is 4 connected squares in one color, then we have 3 connected squares in another color. Next we have two pairs of 2-squares in a 3rd color, and finally four separate squares in a 4th color.

**How to make 4? Imagine trains**

Let's start with the largest rectangle and work down. Can you come up with all the ways to make 4?

Think of attaching cars to make a train. Let's use the rule that order is NOT important. Maybe in a later session we will look at when order is important. But for now we consider these two ways of making 4 the same:

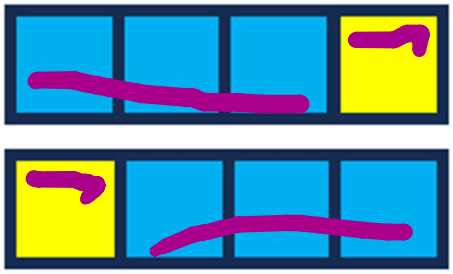
Of course if this was a train they might be two different trains if the train were going right, for example. The engineer would be in the yellow square in the first train and in the end blue one in the second. By the way, you can also use blocks for this exercise.

**Did you get 5 ways to make a 4-across "train"?**

We can use the 4-across rectangle with anything smaller – that's the only way to make 4 with it.

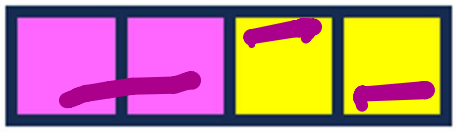
1

How about using the 3-across rectangle and anything smaller? If order doesn't matter then, as we discussed these are the same and either one is a second way to make 4.

2 

Now let's look at using the 2-across rectangles possibly with 1-across ones. When order doesn't matter, there are 2 ways to make 4, right? Here they are, below.

3

 4

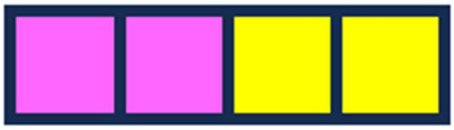
Finally, there's only one way to use only 1-across ones, as below.

5

Let's summarize; here are our 5 ways of breaking 4 to the right.

Thinking in terms of numbers, here are all the ways of making 4:

- **4**
- **3+1**
- **2+2 or 2+1+1**
- **1+1+1+1**



Note how I am organized and start with the biggest number (4) and work my way down. I also don't care about order; 2+1+1 is the same as 1+2+1 and the same as 1+1+2.

**Go ahead and play with the sets of rectangles**

Try to figure out how many ways there are to make all the numbers up to 7. On your own you can continue and go on to 10 or even higher if you want. Please add to this table which I started at 0.

Please note any patterns that you find. Can you predict the count without actually counting? You can make sets of rectangles after the case of 7 or just use numbers. Have fun!

| Number | Ways to break | Count of ways |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |
| 2 | 2; 1+1 | 2 |
| 3 | 3; 2+1; 1+1+1 | 3 |
| 4 | 4; 3+1; 2+2, 2+1+1; 1+1+1+1 you can write this as "4 1s" | 5 |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |

Have Fun!

## Notes for Parents

Happy New Year! Thank you for engaging your child in the kind of math that we do in math circles. I hope that this will be a fun exploratory experience and that your child sees that math is so much more than algorithms.

I decided to start the year with what in mathematics is called partition theory. If you want to further motivate this with your child, ask about what else could be broken up. Perhaps there could be two categories of items, ones that are meant to be reconnected and those that are not.

For example we temporarily break items up when we move a bookshelf from one home to another. We often break it up, separating the shelves and packing up the books. It's easier and safer to move boxed books and separated shelves and then we restore our stocked shelf in the new home.

Your child may not have thought about digestion in this context and our discussion might set the stage for an interesting science lesson, but isn't this an example of permanently breaking something (food)? When we eat we break up our food into a slurry in the mouth so that when we swallow we can aid our digestion. Ultimately food gets broken down so it can nourish us at the cellular level.

I love the work of my friend **James Tanton**. Have you seen his website jamestanton.com? Don't miss his work on "exploding dots". I like his puzzles at globalmathproject.org/nmf-weekly; my daughter wrote in a few years ago and he created and solved a puzzle for her[5] – you can do this for your child too – if you contact him let him know you're my student as he's a friend of mine. James has an excellent video[6] on partitions that you may enjoy showing to your child.

Numberphile is an excellent source for math videos particularly for advanced middle schoolers and above. They have a good video on partitions[7]. The Berkeley Math Circle describes a way of diagraming partitions called Young Diagrams. I may discuss this as well in the future; for now you may enjoy their handout[8]. I teach at many schools including Art of Problem Solving; AoPS has a detailed discussion of partitions[9] that you might enjoy.

There is an OEIS database of sequences. The partition numbers are there as sequence A000041 - oeis.org/A000041. I bet students who haven't seen this collection of sequences before will find it interesting how somebody started naming collections not of seashells or stamps but numbers!

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[5] Check out the issue on math and books that he created based on her inquiry - globalmathproject.org/wp-content/uploads/2020/10/Issue-6_NMF-Weekly-Newsletter_Math-and-Books.pdf.
[6] youtube.com/watch?v=9k9xL9w7-3w
[7] youtube.com/watch?v=NjCIq58rZ8I
[8] mathcircle.berkeley.edu/sites/default/files/handouts/2020/BMC_Adv_Partitions.pdf
[9] artofproblemsolving.com/wiki/index.php/Partition_(combinatorics)

If your child wants to explore making change a bit more and you weren't in our math circle in December, please ask for the last handout of the year. You might get 4 quarters, 10 dimes, 20 nickels, and 100 pennies and let them play with making change. Be sure that your child understands what each coin is worth. Encourage them to explore and come up with questions that you (or I) may not know – why are nickels smaller than quarters but bigger than dimes? Why do some coins have ribbed edges and others don't? Why do we have coins to begin with? Do we know how many coins there are? If you are a coin collector and never spend your coins, what is the intrinsic value of those coins?

Do coins have value in and of themselves or are they defined by an economic context? Does beauty and elegance play a role?

You can bring in history and social justice topics such as discussing the heroic Harriet Tubman and the fact that her $20 bill[10] is due in 2030 (more math – your child can calculate their age then). If you have coins from other countries[11], do a commonality / variability analysis – what's the same? What's different? Any ideas why?

If they want to solve the problem of making change of a dollar, try to solve the problem with the coins. I encourage you to give them the tools then just watch them. If they need help you might suggest coming up with some methodical approach. Maybe they use as many quarters as possible; they will find one solution (4 quarters). Can they do it with 3 quarters? Yes (with 2 dimes and a nickel, 2 dimes and five pennies; 1 dime and 3 nickels, etc.). Don't worry if their patience runs out or if they just think this is difficult – it's okay to have a partial solution.

In general I also recommend that you consider giving your child some cash when you shop. Maybe you go to a farmers' market and give them 12 quarters. Let them help with some purchase decisions perhaps and figure out how to pay and predict change that they will get.

Math circle is supposed to be fun and inspiring. You as the parent knows at what point a little push or nudge may be helpful and when a problem is "too hard" or counterproductive for continued work. I find it very refreshing when I am stuck on a problem to let it go – and come back to it after a break or a night's sleep. Move on to the second problem when the time is right.

The partition problem gets more difficult to enumerate as you get to larger numbers. Going through the table to at least 7 and perhaps higher numbers will reward those who approach this methodically and with organization; this is one big cognitive change from Kindergarten to 1st grade.

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[10] image of bill from June 3, 2021 *Washington Post*, washingtonpost.com/politics/harriet-tubman-20-bill/2021/06/03/62443b5c-bcd1-11eb-9c90-731aff7d9a0d_story.html, accessed Oct. 3, 2023

[11] This image of a 10-Rial coin from Yemen is from numista.com/catalogue/pieces7045.html, accessed Oct. 3, 2023.

Again, being methodical and making predictable choices is very useful and I hope that your child will more and more realize this. As often over the years that I remind my students to show their work, many still don't – but I hope that your child will.

| n | Ways to partition | p(n) |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |
| 2 | 2; 1+1 | 2 |
| 3 | 3; 2+1; 1+1+1 | 3 |
| 4 | 4; 3+1; 2+2, 2+1+1; 1+1+1+1<br>OR: 4; 3+1; 2 2s, 2 and 2 1s; 4 1s | 5 |
| 5 | 5; 4+1; 3+2, 3+1+1; 2+2+1, 2+1+1+1;<br>5 1s | 7 |
| 6 | 6; 5+1; 4+2, 4+1+1; 3+3, 3+2+1,<br>3+1+1+1; 2+2+2, 2+2+1+1, 2+[4 1s];<br>6 1s | 11 |
| 7 | 7; 6+1; 5+2, 5+1+1; 4+3, 4+2+1,<br>4+1+1+1; 3+3+1, 3+2+2, 3+2+1+1,<br>3+[4 1s]; 2+2+2+1, 2+2+1+1+1, 2+[5<br>1s]; 7 1s | 15 |
| 8 | 8; 7+1; 6+2, 6+1+1; 5+3, 5+2+1,<br>5+1+1+1; 4+4, 4+3+1, 4+2+2,<br>4+2+1+1, 4+1+1+1+1; 3+3+2,<br>3+3+1+1, 3+2+2+1, 3+2+1+1+1, 3+[5<br>1s]; 2+2+2+2+2, 2+2+2+1+1, 2+2+[4<br>1s], 2+[6 1s], 8 1s | 22 |
| 9 | 9;<br>8+1;<br>7+2, 7+1+1;<br>6+3, 6+2+1, 6+1+1+1;<br>5+4, 5+3+1, 5+2+2, 5+2+1+1, 5+[4<br>1s];<br>4+4+1, 4+3+2, 4+3+1+1, 4+2+2+1,<br>4+2+1+1+1, 4+[5 1s];<br>3+3+3, 3+3+2+1, 3+3+1+1+1,<br>3+2+2+2, 3+2+[4 1s], 3+[6 1s];<br>[4 2s]+1, [3 2s]+[3 1s], 2+2+[5 1s],<br>2+[7 1s];<br>9 1s | 30 |
| 10 | | 42 |

Let's talk a bit more about partitions. There is a function called the partition function p(n) which takes as its domain n, a non-negative integer. The function p(n) is the number of possible partitions without regard to order. We could have rewritten the table from homework with the first column being n and the last being p(n) as shown to the right with a few more rows filled in for your reference. I did work all the ways out through n=9 but didn't for n=10 in the interest of space – but bet several students will work through that case.

To make it easier to read and see my strategy I spread the case of n=9 out. Note that we have p(9) = 1 + 1 + 2 + 3 + 5 + 6 + 6 + 5 + 1 = 30. These numbers are close to earlier p(n) values … !

These numbers get quite big. Wouldn't it be good to have a way to estimate their value for larger and larger n?

Have you heard of the mathematician Srinivasa Ramanujan? You can read a bit about him in a post[12] by Wolfram Research founder Stephen Wolfram. I highly recommend the 2015 biopic *The Man Who Knew Infinity*[13], though there is a bit of racist violence not appropriate for viewing with younger children; it's a film best for upper middle school and older. He researched this partition function p(n) and, along with his mentor G. H. Hardy, came up with an approximation of p(n) for large n; the theorem is copied here[14].

# THEOREM OF THE DAY

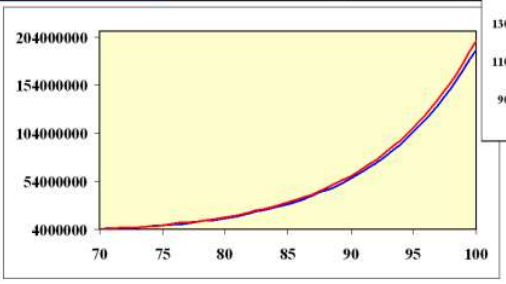
**The Hardy-Ramanujan Asymptotic Partition Formula** *For n a positive integer, let p(n) denote the number of unordered partitions of n, that is, unordered sequences of positive integers which sum to n; then the value of p(n) is given asymptotically by*

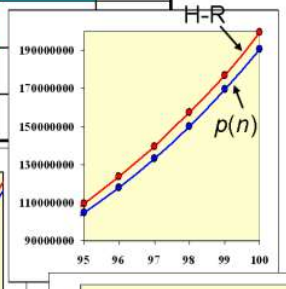
$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\tau\sqrt{n/6}}.$$

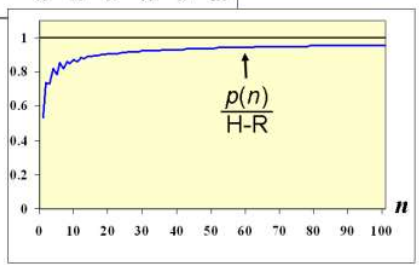
$\tau = 2\pi$ and $e$ in another magical combination!

The value of $p(7)$ is 15, the partitions being displayed in the grid on the right. Note that they are 'unordered', in the sense that 1 + 6, say, is regarded as the same as 6 + 1 and is omitted. By $n = 70$, the number has risen to over 4 million; the Hardy-Ramanujan asymptotic estimate closely shadows this growth (the graphs below centre and right). Exactly how closely, is seen by taking the ratio $p(n)$/H-R (bottom right), the lower curve meeting the line $y = 1$ at infinity.

| | | | | |
|---|---|---|---|---|
| 7 | 6+1 | 5+2 | 5+1+1 | 4+3 |
| 4+2+1 | 4+1+1+1 | 3+2+2 | 3+3+1 | |
| 3+2+1+1 | 3+1+1+1+1 | 2+2+2+1 | | |
| 2+2+1+1+1 | 2+1+1+1+1+1 | | | |
| 1+1+1+1+1+1+1 | | | | |




Srinivasa Ramanujan (1887–1920) had already made deep discoveries about the partition function when he was himself 'discovered' by G.H. Hardy in 1913. Five years later, despite Ramanujan being already terminally ill, they published this remarkable asymptotic formula, derived from a series which yields the *exact value* of $p(n)$ in $\sqrt{n}$ terms. It was discovered independently, in 1920, by James Victor Uspensky.

**Web link:** www.ias.ac.in/describe/article/reso/001/12/0081-0091: Atle Selberg talks about Ramanujan's life and work; his study of $p(n)$ is discussed on pp. 86–87.

**Further reading:** *Integer Partitions, 2nd revised ed.* by George E. Andrews and Kimmo Eriksson, Cambridge University Press, 2004.

Created by Robin Whitty for www.theoremoftheday.org

Thanks for coming to math circle - best wishes!

Dilip

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[12] writings.stephenwolfram.com/2016/04/who-was-ramanujan, accessed Jan. 17, 2024

[13] The image promoting the film is from imdb.com/title/tt0787524, accessed Jan. 17, 2024.

[14] copied from theoremoftheday.org/NumberTheory/Partitions/TotDPartitions.pdf, accessed Jan. 16, 2024