Invariants [1]

Warm-ups

1. A code uses only the letters A and O, and satisfies the following properties:

   (a) If you delete two neighboring letters AO from any code, or insert them, then you get a code with the same meaning.

   (b) If you delete two neighboring letters OA from any code, or insert them, then you get a code with the same meaning.

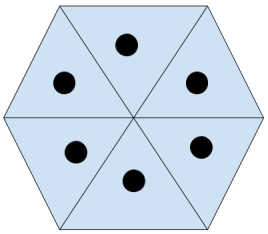
   (c) If you delete four neighboring letters AAOO from any code, or insert them, then you get a code with the same meaning.

   Do the codes AAO and OAA have the same meaning, or different meanings, or is there not enough information to be sure?
   What about the codes AAO and OOA?

2. There are 7 plastic cups on a table, all standing upside down. You are allowed to turn over any 4 of them in one move. Is it possible to get all the cups standing up?

3. There are six sparrows sitting on six trees, one sparrow in each tree. The trees stand in a row, with 10 meters between any two neighboring trees. If a sparrow flies from one tree to another, then at the same time, some other sparrow flies from some tree to another the same distance away, but in the opposite direction. Is it possible for all the sparrows to gather on one treee? What if there are seven trees and seven sparrows?

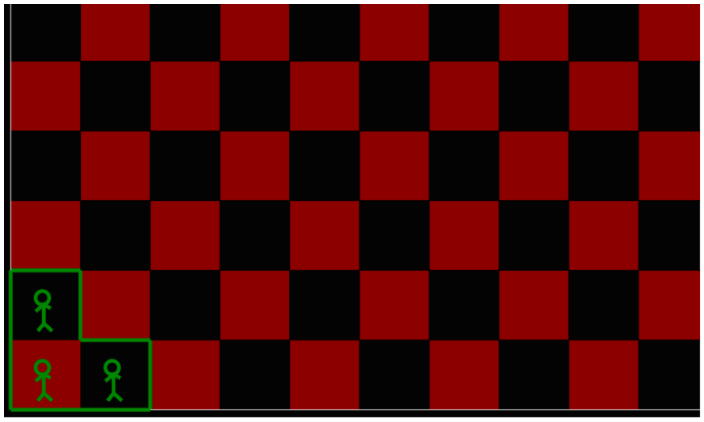
4. A game is played on a circle divided into 6 sectors, as shown. To start, a stone is placed in each sector. On your turn, you can pick up any two stones, and move each stone to a sector that borders its original position. You win the game if you can move all the stones into a single sector. Give a winning strategy or show that it is not possible to win.



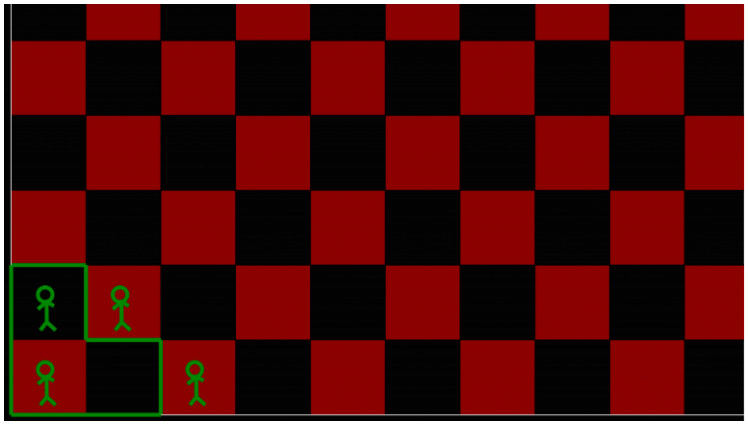
[1] Problems from Mathematical Circle (Russian Experience)

Free the Clones

5. Imagine that you are standing at the corner of an infinite checkerboard. Three clones are imprisoned in the three tiles at the corner of the board.



The only way for a clone to move is to split itself into two clones, which will occupy the tiles above and to the right of the original clone, as shown below. Only one clone can occupy a tile at a time. For example, if the clone on the right were the first to split, its move would look like this:



Is it possible to free all the clones from the prison?

See https://openprocessing.org/sketch/126042/ for an app of this game.

See https://www.mathteacherscircle.org/news/mtc-magazine/ws2017/freeing-the-clones/ for a spoiler solution ... but please not until after class!

Coloring Checkerboards

6. In an 8 x 8 grid, one of the boxes is colored black and all the others are white. Prove that it is not possible to make all the boxes white by "recoloring" the rows the columns. "Recoloring" means changing the color of all the boxes in a single row or column.

7. Solve the same problem for a 3 x 3 grid if initially there is only one black box in a corner of the table.

8. Solve the same problem for an 8 x 8 table if initially all four corner boxes are black and all the others are white.

Chameleons

9. There are 11 green, 13 brown, and 16 red chameleons on an island. When two chameleons of different colors meet, they both change their colors to the third color. For example, if a green and a red chameleon meet, they both become brown. Is it possible for all chameleons to be the same color?

10. Same problem, but on a different island where there are 13 green, 15 brown, and 17 red chameleons to start.

Miscellaneous

11. Dr. Gizmo has invented a coin changing machine which can be used in any country in the world. No matter what the system of coinage, the machine takes any coin, and, if possible, returns exactly 5 coins with the same total value. Prove that no matter how the coinage system works in a given country, you can never start with a single coin and end up with 26 coins.

12. The numbers 1, 2, 3, ... 2024 are written on a blackboard. We are allowed to erase any two of them and replace them with their difference. Is it possible to get to a situation in which all the numbers are zeros? What if we start with the numbers 1, 2, 3, ... 2025?

13. There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than 1 stone, throw away one stone from that heap, and divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones?