Egyptian Fractions

Chapel Hill Math Circle: Advanced Group

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1 Introduction

An Egyptian Fraction is a sum of positive unit fractions (fractions with 1 in the numerator, and a positive integer in the denominator). The Egyptians first did many calculations and kept records using these types of fractions. They also had special symbols for $\frac{2}{3}$ and $\frac{3}{4}$, but we will only consider the unit Egyptian fractions.

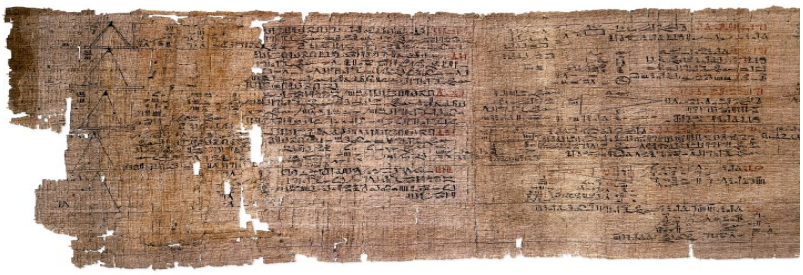

Figure 1: The Rhind Papyrus, purchased by Alexander Rhind in 1858 is dated to around 1650 BCE [RP1]

We will explore Egyptian Fraction representations of the number 1 (often called "unity"), counting such representations, how to construct representations, and more!

2 Building Blocks

Definition 2.1. A *Unit Fraction* is a fraction with 1 in the numerator, and a positive integer in the denominator.

Question 2.1. List 3 different unit fractions.

Definition 2.2. An *Egyptian Fraction Representation of One* is any sum of unit fractions that adds up to 1.

Question 2.2. List 3 different Egyptian Fraction Representations of One.

Notice that in our definition of an Egyptian Fraction Representation of 1, we don't actually put any restriction on how many fractions we can use. We could use infinitely many if we wanted! One of the most famous infinite Egyptian Fraction Representations of 1 is

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

Question 2.3. (Tricky) Can you think of another infinite Egyptian Fraction Representation of 1?

Let's turn our attention back to finite sums.

Definition 2.3. An Egyptian Fraction Representation of 1 that uses exactly n fractions is called an *Egyptian Fraction Representation of 1 in n terms.*

Question 2.4. Can you list the only (1) Egyptian Fraction Representation of 1 using 1 term?

Question 2.5. Can you list the only (1) Egyptian Fraction Representation of 1 using 2 terms?

In order to count the number of Egyptian Fraction Representations of 1 with more than 2 terms, we need to talk about the order of the terms.

Definition 2.4. An *Ordered Egyptian Fraction Representation of 1 in n terms* is an Egyptian Fraction representation of 1 in n terms, with the denominators are listed in increasing order. If the denominators are not listed in increasing order, we call this an *Unordered Egyptian Fraction Representation of 1 in n terms.*

Example 2.1. The Egyptian Fraction Representation of 1 on the left is ordered, while the representation on the right is unordered.

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \qquad\qquad 1 = \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$$

Question 2.6. Can you list the 3 Ordered Egyptian Fraction Representation of 1 using 3 terms? How many Unordered Egyptian Fraction Representations of 1 in 3 terms must there be?

I think we can all agree, it's getting really old writing all of these fractions. So instead of writing the fractions over and over again, let's get some new notation to make things easier to write down.

Definition 2.5. Let $1 = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}$ be an Egyptian Fraction Representation of 1 in n terms. We will now use the notation $(x_1, x_2, x_3, \ldots, x_n)$ to mean the same thing, that is, instead of listing out the fractions and adding them together, we will just list out the denominators in the fractions as a list.

Example 2.2. The 3 Ordered Egyptian Fraction Representations of 1 in 3 terms are $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$ in our new notation.

3 Splittings

Let us look and see if there are any relationships between the Egyptian Fraction Representations of 1 in 2 terms and those in 3 terms:

The only Egyptian Fraction Representation of 1 in 2 terms is $(2, 2)$ and the 3 Ordered Representations of 1 in 3 terms are $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$.

Question 3.1. Can you think of a way to "split" $(2, 2)$ to get any of the other 3 representations in 3 terms?

It may be easier to see the splittings in the fraction form:

Example 3.1. Let's start with $(2, 2)$ and manipulate it so that we can one of the representations in 3 terms.

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{2}(1) \\ &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \end{aligned}$$

We were able to split one of the copies of $\frac{1}{2}$ into two copies of $\frac{1}{4}$!

Question 3.2. (Tricky) After seeing the above example, can you think of a way to split $(2, 2)$ into $(2, 3, 6)$?

In order to split $(2, 2)$ into $(2, 3, 6)$, notice how we split $(2, 2)$ into $(2, 4, 4)$. We split one copy of $\frac{1}{2}$ into two pieces, each piece had size one-half of the whole $\frac{1}{2}$. Is there a way we could have split one of the copies of $\frac{1}{2}$ into two pieces that were different sizes?

Example 3.2. Yes! Take $(2, 2)$ again:

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{2}(1) \\ &= \frac{1}{2} + \frac{1}{2}\left(\frac{2}{3} + \frac{1}{3}\right) \\ &= \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{6} \end{aligned}$$

Question 3.3. (Super Tricky) After seeing the above examples, can you think of a way to split $(2, 2)$ into $(3, 3, 3)$?

So how did we know to split up the copy of $\frac{1}{2}$ the two different ways that we did? Notice that in the first example, we multiplied by 1, which we split up as $\frac{1}{2} + \frac{1}{2}$. In the second example, we multiplied by 1, which we split up as $\frac{2}{3} + \frac{1}{3}$. The trick to notice here is that (unbelievably) $1 + 1 = 2$ and $2 + 1 = 3$! That is, in the first example, the numerators of how we split up the 1 were 1 and 1, and the denominator of both pieces was $1 + 1 = 2$. In the second example, the numerators of how we split up the 1 were 2 and 1, and the denominator of both pieces was $2 + 1 = 3$.

Question 3.4. Do you have any ideas where we got the numerators 1 and 1 in the first example? Do you have any ideas where we got the numerators 2 and 1 in the second example? *Hint: What fraction are we trying to split?*

This leads us to a Theorem!

Theorem 3.1. *Let $\frac{p}{q}$ be a fraction in simplest terms. Then we can write*

$$\frac{p}{q} = \frac{1}{a} + \frac{1}{b}$$

if and only if we can find two other natural numbers c and d that are divisors of q and p divides $c + d$. The correspondence is given by

$$a = \frac{q(c+d)}{pd} \qquad\qquad b = \frac{q(c+d)}{pc}$$

Let's see the theorem in action and how to use it.

Example 3.3. Suppose we want to write $\frac{5}{12}$ as the sum of two unit fractions. Then our theorem above says that if we can find two numbers c and d that divide 12 (the denominator), and such that 5 (the numerator) divides $c + d$, then we know we can set

$$a = \frac{12(c+d)}{5d}$$

and

$$b = \frac{12(c+d)}{5c},$$

and then we will have that

$$\frac{5}{12} = \frac{1}{a} + \frac{1}{b}.$$

So let's start looking for c and d.

- Now c and d both have to be divisors of 12, so let's look at the divisors of 12. The divisors of 12 are $\{1, 2, 3, 4, 6, 12\}$. So c and d are going to both be numbers from this list.
- We also need 5 to divide $c + d$. So let's just look through all the pairs of divisors and see whether their sums are divisible by 5!
- But do we really need to do ALL of the pairs of divisors? NO! If we look at the pair $c = 4$ and $d = 6$, that will give us the values $a = 4$ and $b = 6$... but if we look at the pair $c = 6$ and $d = 4$, that will give us the values $a = 6$ and $b = 4$! These are the same things just switched! So we only need to worry about pairs with $c \leq d$. Complete the table below.

c	d	$c+d$	$5\vert(c+d)$?
1	1	2	NO
1	2	3	NO
1	3	4	NO
1	4	5	YES
1	6		
1	12		
2	2		
2	3		
2	4		
2	6		
2	12		
3	3		
3	4		
3	6		
3	12		
4	4		
4	6		
4	12		
6	6		
6	12		
12	12		

Thus, we see that our pairs of divisors of 12 that work are:

c	d
1	4
2	3
3	12
4	6

- Now for each pair of choices for c and d, we can compute the corresponding a and b from above (complete the table below):

c	d	$a = \frac{12(c+d)}{5d}$	$b = \frac{12(c+d)}{5c}$
1	4	3	12
2	3		
3	12		
4	6		

- Do you notice anything about the values of a and b we got for the different values of c and d? Think about why the values $c = 1$ and $d = 4$ gives the same values of a and b as for the choices of $c = 3$ and $d = 12$.

- Because $c = 3$ and $d = 12$ are not *relatively prime*, we know that there are other values of c and d that are smaller that give the same a and b values, like $c = 1$ and $d = 4$!

Definition 3.1. Two numbers x and y are said to be *relatively prime* if $\gcd(x, y) = 1$.

Remark 3.1.1. Note that two numbers being relatively prime does not mean that either of them are actually *prime*. For example $x = 14$ and $y = 15$ are relatively prime, but neither of them are prime numbers.

So we have learned that we can write $\frac{5}{12}$ as the sum of two unit fractions in the following two ways:

$$\frac{5}{12} = \frac{1}{3} + \frac{1}{12}$$
$$\frac{5}{12} = \frac{1}{4} + \frac{1}{6}$$

Moving back to splittings, we can now use this theorem to split all of representations of 1 in 3 terms into representations of 1 into 4 terms!

Question 3.5. How many Ordered Egyptian Fraction Representations of 1 in 4 terms are there? What are they?

Question 3.6. (Challenge) How many Ordered Egyptian Fraction Representations of 1 in 5 terms are there? What are they?

4 Greedy Algorithm

Question 4.1. Does every positive rational number (fraction) have an Egyptian Fraction Representation?

Example 4.1.

$$\frac{25}{14} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{28}$$

Question 4.2. In this example, why did we start with $\frac{1}{1}$?

Note that once we "use" $\frac{1}{1}$ we have:

$$\frac{25}{14} = \frac{1}{1} + \frac{11}{14}$$

So now we've reduced the question of whether $\frac{25}{14}$ has an Egyptian Fraction Representation to whether $\frac{11}{14}$ has an Egyptian Fraction Representation!

Question 4.3. Continuing with the same expansion as above, why did we use $\frac{1}{2}$ as the next fraction? Did we have any choice for our next fraction after using $\frac{1}{2}$?

A useful function when talking about Egyptian Fraction Representations is the floor function.

Definition 4.1. Let x be any real number. We define the *Floor of* x, denoted $\lfloor x \rfloor$, to be the largest integer that is less than or equal to x.

Example 4.2.

- $\lfloor 2.7 \rfloor = 2$.
- $\lfloor 8 \rfloor = 8$.
- $\lfloor \sqrt{2} \rfloor = 1$.

Question 4.4. What are is the value of $\lfloor x \rfloor$ for the following values of x?

- $x = 0$

- $x = -\pi$
- $x = \frac{11}{10}$

Another very useful theorem will lead to our Greedy Algorithm:

Theorem 4.1. *Every positive rational number $\frac{p}{q}$ can be written as*

$$\frac{p}{q} = \frac{1}{x} + r,$$

where x is one more than $\frac{q}{p}$ rounded down to the nearest integer, and r is a rational number strictly smaller than $\frac{p}{q}$. That is $x = \left\lfloor \frac{q}{p} \right\rfloor + 1$ and $r < \frac{p}{q}$.

Let's see this theorem in action:

Example 4.3. Write $\frac{5}{7}$ using an Egyptian Fraction Representation.

- First, we note that $\left\lfloor \frac{7}{5} \right\rfloor + 1 = 2$.
- Next, we subtract $\frac{1}{2}$ from $\frac{5}{7}$:

$$\frac{5}{7} - \frac{1}{2} = \frac{3}{14}$$

- Now we repeat the process. First we note that $\left\lfloor \frac{14}{3} \right\rfloor + 1 = 5$.
- Nex we subtract $\frac{1}{5}$ from $\frac{3}{14}$:

$$\frac{3}{14} - \frac{1}{5} = \frac{1}{70}$$

- Since the last difference we computed was itself a unit fraction, we're done! Now we can put the pieces back together and we get:

$$\frac{5}{7} = \frac{1}{2} + \frac{1}{5} + \frac{1}{70}$$

Definition 4.2. A *Greedy Algorithm* is any algorithm (set of instructions/process) that follows the pattern of making the biggest/most-optimal/greediest choice at any step.

Question 4.5. Why is the theorem above basically a Greedy Algorithm?

Remark 4.1.1. Notice that every time in the process we used above, we took the *largest* fraction we could away from the initial fraction we started with.

Algorithm 1 Greedy Egyptian Fraction Representation Algorithm

Greedy(p,q,n,ℓ)

Input: p - the numerator of $\frac{p}{q}$ in simplest form, q - the denominator of $\frac{p}{q}$ in simplest form, n the number of terms to use, ℓ - the denominator of the last used fraction.

Output: Count - the number of Egyptian Fraction Representations of $\frac{p}{q}$ in n terms, with every denominator at east as big as ℓ.

```
Count ← 0
if n is 1 then
    if q is not divisible by p OR q is less than p × ℓ then
        Count ← Count + 0 (no representations)
    else
        Count ← Count + 1 (one representation)
else
    bdenom ← ⌊nq/p⌋
    if q is less than p × ℓ then
        adenom ← ℓ
    else
        adenom ← ⌊q/p⌋ + 1
    for adenom ≤ i ≤ bdenom do
        Count ← Count + Greedy(ip − q, iq, n − 1, i)
return Count
```

Remark 4.1.2. The algorithm above can be modified to also keep track of and store the actual representations, not just the count of how many representations there are.

Question 4.6. In what order does the Greedy Algorithm find the 3 different Ordered Egyptian Fraction Representations of 1 in 3 terms?

Question 4.7. In what order does the Greedy Algorithm find the 14 different Ordered Egyptian Fraction Representations of 1 in 4 terms?

5 Distinct Terms

Question 5.1. Given a positive rational number r, does r have a representation as an Egyptian Fraction with the denominators in the the representation all being distinct?

Question 5.2. How can we use our greedy algorithm to find an Egyptian Fraction Representation of 1 for any number of terms?

Question 5.3. Look at the the Egyptian Fraction Representations of 1 that have the largest denominators for each number of terms. For $n = 1$ the only representation is (1). For $n = 2$ the only representation is $(2, 2)$. For $n = 3$ the representation with the largest denominator is $(2, 3, 6)$. For $n = 4$ the only representation is $(2, 3, 7, 42)$. Can you find the solution with the largest denominator for 5 terms?

Definition 5.1. Let n be a natural number. We define the Ordered Egyptian Fraction Representation of 1 in n terms with the largest denominator to be the *Sylvester Representation of 1 in n terms*.

Question 5.4. Let $n \geq 3$ be a natural number. Can you find a pattern for every denominator in the Sylvester Representation of 1 in n terms (except for the last denominator), in terms of the previous terms in the Sylvester Representation of 1 in n terms?

Question 5.5. Let $n \geq 3$ be a natural number. Can you find a pattern for the last denominator in the Sylvester Representation of 1 in n terms, in terms of the previous terms in the

Sylvester Representation of 1 in n terms?

Question 5.6. Same question as above, but can you find a recursive pattern that depends only on the next-to-last term?

Remark 5.0.3. The last denominators in each of the Sylvester Representations forms a sequence: $\{1, 2, 6, 42, 1806, 3263442, \ldots\}$. It turns out [A076393] that there is a a number $E \approx 1.26408473530530\ldots$ such that the n^{th} term in the above sequence, which we will denote s_n, is given by

$$s_n = \left\lfloor E^{2^{n+1}} + \frac{1}{2} \right\rfloor - 1.$$

This shows us that s_n grows even faster than an exponential function, it grows DOUBLY-Exponentially!

References

[A076393] OEIS Foundation Inc. (2019), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A076393`.

[RP1] Rhind Papyrus Image Wikipedia - Paul James Cowie (Pjamescowie) - `https://commons.wikimedia.org/wiki/File:Rhind_Mathematical_Papyrus.jpg`