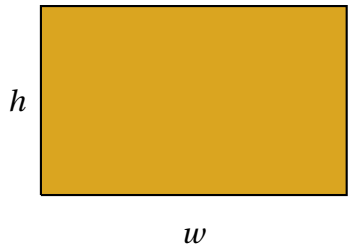

Figure 0.1: A golden rectangle with $\frac{w}{h} = \varphi$.

The Golden Ratio

Abstract

In this session, we shall explore the *golden ratio*, denoted by the Greek letter φ. We first present some of the historical context for interest in φ, then use geometry and algebra to compute its value explicitly. We next see how φ appears throughout mathematics, from a regular pentagon to the *Fibonacci sequence* to its *(simple) continued fraction* representation. We shall also explore related quantities, including the *metallic ratios* and the *plastic ratio*.

Background needed: Prerequisites include include algebra, including the *quadratic formula*. Some geometry, including *similar polygons*, will also be useful. Familiarity with the aforementioned Fibonacci Sequence will be helpful, but it will be presented here in a self-contained way.

0 Warmup

Our goal is to study properties of the ratio $\frac{w}{h}$ of the side lengths in the rectangle of the shape given in Figure 0.1. As prerequisites for our session, it will help to answer the following first. You do *not* need to know these answers already. Many questions will be revisited later in the worksheet.

Exercises:

0.1 What is *the golden ratio* (denoted by φ, the Greek letter phi[1])? What is its value, and what does this ratio represent geometrically?

[1] Phi is typically pronounced "fee" or sometimes "fie". You may also see Φ, or capital-phi, to denote the golden ratio.

0.2 What is the *Fibonacci sequence*? Are you aware of any relationship between φ and the Fibonacci sequence?

Remark. The advanced group of Chapel Hill Math Circle explored the Fibonacci Sequence in our sessions of February 11 and February 25, 2023.

0.3 What are *similar polygons*? For example, if $\triangle ABC$ is similar to triangle $\triangle DEF$, denoted[2] $\triangle ABC \sim \triangle DEF$, then what can we conclude?

0.4 What is a *quadratic equation*? What is the *quadratic formula*?

0.5 Let n be a positive integer such that $\sqrt{n}$ is an irrational number. If a, b, c, d are integers, and if

$$x := \frac{a + b\sqrt{n}}{c + d\sqrt{n}}, \tag{0.1}$$

then what does it mean to *rationalize* x? What is the method for rationalizing such a number?

[2] *Note:* Be careful about the order here! Saying $\triangle ABC \sim \triangle DEF$ is *different* from saying $\triangle ABC \sim \triangle DFE$. For similarity, we are interested not simply in whether one polygon is similar to another, but also the *order* of how the corresponding vertices of the two pair up.

0.6 What is *mathematical induction*?

0.7 What is *proof by contradiction*?

1 An Introduction to the Golden Ratio, φ

Let us begin by defining the golden ratio.

Definition 1.1 (The Golden Ratio, φ)**.** Let w, h be the side lengths of a rectangle in Figure 1.1, where $w > h$. Suppose that for such a rectangle, we have

$$\frac{w}{h} = \frac{h}{w-h}. \tag{1.1}$$

so that the red rectangle is similar to the combined blue-and-red rectangle. Then the common ratio in (1.1) is defined to be *the golden ratio*, denoted by the Greek letter φ.

As we shall see below, φ can be computed explicitly, and as an approximation, $\varphi \approx 1.618$.

This ratio has been called "golden" for aesthetic reasons, and the proportion (or at least approximations thereto) have been identified in art, architecture, music, and nature. However, many such purported connections to φ are spurious, even just to rouch approximation. For example, see [2], which discusses φ in regards to the construction of The Great Pyramid of Giza, The Parthenon in Athens, the works of Leonardo da Vinci, its appearance as ideal proportions of the human body, and other questionable examples.

For our purposes, we will be interested not in these subjective evaluations of beauty, let alone the numerology of φ by mystics and others. Instead, our primary interest will be mathematical properties of φ and its connections elsewhere. Some examples of φ in nature are supported, though, and we shall also mention some from biology.

Exercises:

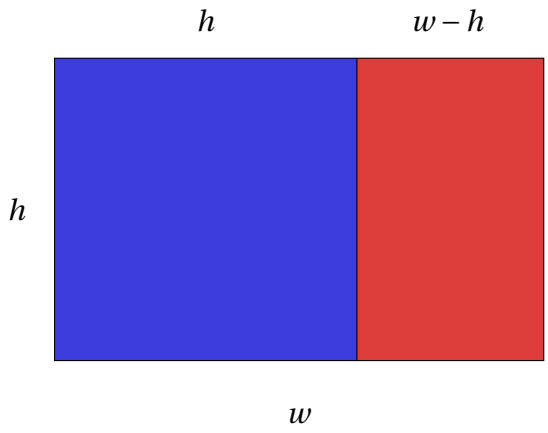

Figure 1.1: If removing the blue square leaves a red rectangle similar to the original, then $w/h = \varphi$.

1.1 Let w, h be as in Definition 1.1. Prove that if $\varphi := \frac{w}{h}$, then (1.1) can be written in the form

$$\varphi = \frac{1}{\varphi - 1}. \tag{1.2}$$

1.2 Show that

$$\varphi^2 = \varphi + 1. \tag{1.3}$$

1.3 Using (1.2) or (1.3), compute φ explicitly using the quadratic formula. In other words, express φ in the form of a solution to a quadratic formula, *not* as a decimal approximation to φ.

Remark. In the quadratic formula, in general there are two possible solutions to a quadratic equation. How can you tell which solution is our desired ratio φ and which is not?

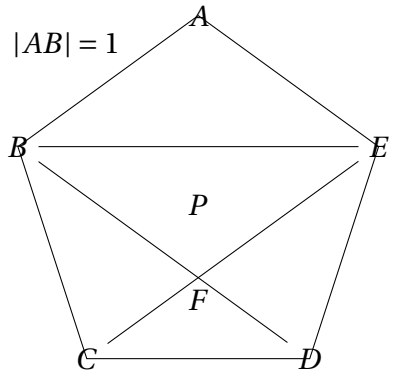

Figure 1.2: A regular pentagon $ABCDE$ with side length 1.

1.4 Let P be a regular pentagon with side length 1, as in Figure 1.2. What is the length of the side of a diagonal of P? That is, in the regular pentagon below where $|AB| = |BC| = |CD| = |DE| = |EA| = 1$, what is the length $|BE|$?

Hint: Let F be the point at the intersection of $\overline{BD}$ and $\overline{CE}$. What can you say about $\triangle ABE$ and $\triangle BFE$? What can you say about $\triangle BFE$ and $\triangle CFD$? What can you conclude?

Remark. This application of φ to dimensions in a regular pentagon is inspired by [7], though more connections to φ and regular pentagons and pentagrams can also be found in [3].

1.5 From Exercise #1.4, assuming you are familiar with trigonometry, compute $\cos 36^\circ = \sin 54^\circ$.

2 Properties of φ

Exercises:

2.1 We learned in Section 1 that φ satisfies the equation $\varphi^2 = \varphi + 1$. Using this show that $\varphi = 1 + \frac{1}{\varphi}$.

Conclude that

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{\varphi}} \tag{2.1}$$

2.2 Assuming we can generalize (2.1) infinitely,[3] explain why

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}}, \tag{2.2}$$

Remark. The expression on the right-hand side of (2.2) is called the *infinite simple continued fraction* for φ, The advanced group of Chapel Hill Math Circle has explored continued fractions before, most recently in our sessions from October 16, 2021 and October 30, 2021.

2.3 Using (2.2) alone—i.e., *without* using the explicit value for φ computed above—explain why φ is an irrational number.

[3] Such assumptions *are* mathematically valid, but justifying them rigorously would require too long and technical a digression for our purposes.

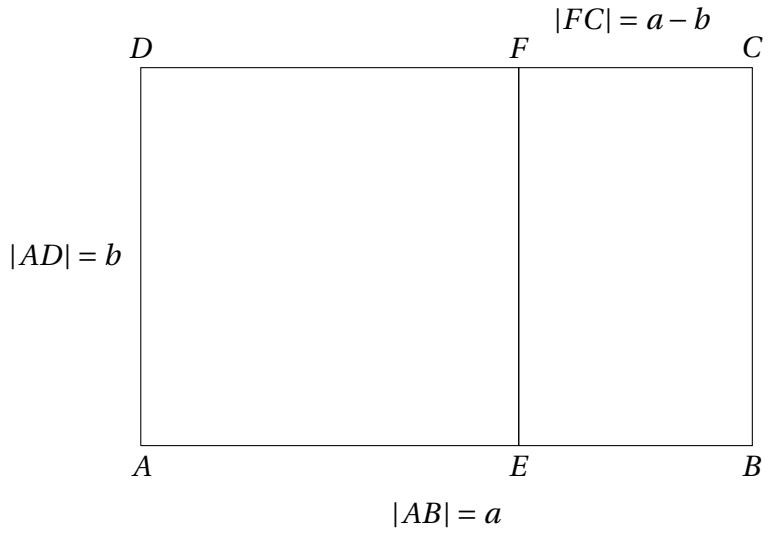

Figure 2.1: Proving the irrationality of φ via geometry.

Hint: If φ were a rational number, would we be able to form an *infinite* simple continued fraction for φ?

2.4 We next reprove that φ is irrational geometrically rather than algebraically. Proceeding via *proof by contradiction*, *assume* instead that φ is a rational number, $\varphi = \frac{a}{b}$ where a, b are positive integers, and φ is in lowest terms. (In other words, we must have $\gcd(a,b) = 1$.)

Form an $a+b$-by-a rectangle, as in Figure 2.1.

(a) Given that $\frac{a}{b} = \varphi$, prove that $\frac{b}{a-b} = \varphi$.

(b) Explain why $\frac{b}{a-b}$ is a "smaller" fraction expression for φ than $\frac{a}{b}$.

(c) Explain why this is a contradiction, and conclude that φ is therefore irrational.

2.5 Compute $\frac{1}{\varphi}$ explicitly. Rationalize your answer in the context of Exercise #0.5.

Remark. In case you have not already computed φ explicitly to solve Exercise #1.3, you may use that

$$\varphi = \frac{1+\sqrt{5}}{2}. \tag{2.3}$$

3 An Introduction to the Fibonacci Sequence

We asked about the Fibonacci sequence above in Exercise #0.2, and it is a mathematical object that is closely related to φ.

Definition 3.1. The *Fibonacci Sequence* is the sequence $(F_n) = F_1, F_2, F_3, \ldots$ defined recursively via

$$F_1 := 1, \tag{3.1}$$
$$F_2 := 1, \text{ and} \tag{3.2}$$
$$F_{n+2} := F_{n+1} + F_n \text{ for all } n \geq 1. \tag{3.3}$$

That is, the Fibonacci Sequence begins

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots,$$

and each successive term is the sum of the two previous terms. Namely: $1+1=2$, $1+2=3$, $2+3=5$, etc.

Exercises:

3.1 What are the next two terms of the Fibonacci Sequence, F_{10} and F_{11}?

3.2 *Assuming* $\lim_{n\to\infty} F_{n+1}/F_n$ exists, use the recursive formula to show this limit is φ.

Hint: If the limit exists, then we must have

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}. \tag{3.4}$$

With (3.4), what can you say after dividing both sides of (3.3) by F_{n+1}?

3.3 From (1.3), we have that $\varphi^2 = \varphi + 1$, meaning that we can write φ^2 in terms of lower powers of φ. We now generalize this to higher powers of φ.

First, express φ^3 in the form $\varphi^3 = a\varphi + b$, where a, b are integers.

3.4 Repeat the process of Exercise #3.3 for φ^4, φ^5, and φ^6. Do you notice any patterns?

3.5 For every positive integer n, express φ^n in the form $a\varphi + b$, where a, b are integers depending on n.

Note: Proving that your formula works may involve a proof technique such as mathematical induction.

3.6 Recall that φ satisfies the quadratic equation $x^2 - x - 1 = 0$. This equation has a second root, which we shall denote by ψ. Explicitly, we have

$$\psi = -\frac{1}{\varphi} = \frac{1-\sqrt{5}}{2}. \tag{3.5}$$

Show that an analogous formula to that of 3.5 holds for ψ.

For every positive integer n, express ψ^n in the form $a\psi + b$, where a, b are integers depending on n.

3.7 Using the above, derive *Binet's Formula* for F_n, a closed-form expression for the nth Fibonacci number: for every positive integer n,

$$F_n = \frac{1}{\sqrt{5}} \cdot \left(\varphi^n - \psi^n\right). \tag{3.6}$$

Hint: From Exercises #3.5–3.6, you have two equations. Can you eliminate the F_{n-1} term? If so, how does this help?

Note: This derivation is taken from [3]. Further, note that each of the component terms in Binet's Formula is irrational, but the value of each F_n is not just rational but an integer!

4 Generalizations to the Golden Ratio: Metallic Ratios

In Definition 1.1 via geometry, we defined the golden ratio to be the unique *aspect ratio* for a rectangle such that if we remove a square whose side length is that of the shorter side of our rectangle, the remaining rectangle will have the same aspect ratio as the original. With this picture in mind, we consider a modification to that definition:

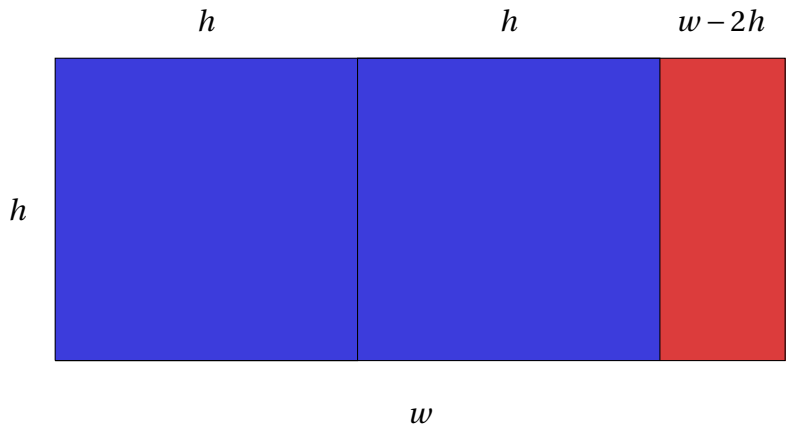

Figure 4.1: If removing the *two* blue square leaves a red rectangle similar to the original, then $w/h = \sigma$, the silver ratio.

Definition 4.1 (The Silver Ratio, σ)**.** Let w, h be the side lengths of a rectangle in Figure 1.1, where $w > h$. Suppose that for such a rectangle, we have

$$\frac{w}{h} = \frac{h}{w-2h}. \tag{4.1}$$

so that the red rectangle is similar to the combined blue-and-blue-and-red rectangle. Then the common ratio in (4.1) is defined to be *the silver ratio*, denoted by[4] σ, the Greek letter sigma.

As we shall see below, σ can be computed explicitly, and as an approximation, $\sigma \approx 2.414$.

As the golden ratio is related to the Fibonacci Sequence, we introduce a sequence similarly related to the silver ratio:

Definition 4.2. The *Pell Sequence* is the sequence $(P_n) = P_1, P_2, P_3, \ldots$ defined recursively via

$$P_1 := 1, \tag{4.2}$$
$$P_2 := 2, \text{ and} \tag{4.3}$$
$$P_{n+2} := 2P_{n+1} + P_n \text{ for all } n \geq 1. \tag{4.4}$$

That is, the Pell Sequence begins

$$1, 2, 5, 12, 29, 70, 169, \ldots,$$

and each successive term satisfies the recursion formula in (4.4). Namely: $2 \cdot 2 + 1 = 5$, $2 \cdot 5 + 2 = 10$, $2 \cdot 12 + 5 = 29$, etc.

Exercises:

[4]The silver ratio has different common notations, from σ (the Greek letter sigma) to λ (the Greek letter lambda) to δ_S (where δ is the Greek letter delta). We shall use σ here.

4.1 Analogously with (1.2) and (1.3) above, prove that

$$\sigma = \frac{1}{\sigma - 2} \tag{4.5}$$

and

$$\sigma^2 = 2\sigma + 1. \tag{4.6}$$

Use one or more of these to compute σ precisely.

4.2 Conclude that

$$\sigma = 2 + \frac{1}{\sigma} \tag{4.7}$$

4.3 Generalizing (4.7) infinitely, explain why

$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}}}, \tag{4.8}$$

The expression on the right-hand side of (4.8) is the infinite simple continued fraction for σ,

4.4 Using (4.8) alone—i.e., *without* using any explicit value for σ computed above—explain why σ is an irrational number.

4.5 What are the next two Pell numbers, P_8 and P_9?

4.6 Can you generalize other results from previous sections for the silver ratio and the Pell Sequence? For example, in Section 3, we were able to produce Binet's Formula, a closed-form expression for the nth Fibonacci number F_n in terms of n alone. Can you do something similar here for the Pell sequence?

4.7 *Assuming* $\lim_{n\to\infty} P_{n+1}/P_n$ exists, prove this limit is the silver ratio.

4.8 In Section 2, we saw how the golden ratio arose as the diagonal of a regular pentagon. Can you generalize this to find a similar result between the silver ratio and some other regular polygon?

Note: for more on the silver ratio, see [6].

4.9 We can further generalize the silver ratio to an entire family of named ratios:

Definition 4.3 (The Metallic Ratios, S_N)**.** Let n be a fixed positive integer. Further, assume w, h be the side lengths of a rectangle, where $w > h$. Suppose that for such a rectangle, we have

$$\frac{w}{h} = \frac{h}{w - Nh}. \tag{4.9}$$

so that after removing N copies of the square of side length h, the remaining rectangle is similar to the original larger rectangle. Then the common ratio in (4.9) is defined to be *the metallic ratio of order N*, denoted by[5] S_N.

Remark. Using the more general notation in Definition 4.9, we see that

$$\varphi := S_1, \text{ and}$$
$$\sigma := S_2.$$

Generalizing our results for the golden and silver ratios, find a quadratic equation for S_n. Use that to express S_n as an infinite simple continued fraction, show that each S_n is irrational, and compute S_n explicitly.

4.10 Fixing N, How would you define a general sequence (G_n), relative to N, that generalizes the roles of the Fibonacci Sequence and Pell Sequence relative to the golden and silver ratios, respectively? For such a sequence, what could you say about the limit

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n}$$

as a function of N?

[5] Notation for the nth order metallic mean may not be standardized. We are following the choice made by Wikipedia in the article linked in our definition.

Separately, there are even more general notions of metallic ratios. For one example, see [8]. There, the general metallic mean of order (p, q) is defined relative the positive root of the quadratic equation $x^2 = px + q$.

4.11 Can you find a closed-form formula for G_n in terms of n (and N) alone?

4.12 **Open Question:** Can you generalize the connections between the golden and silver ratios and regular polygons to the metallic ratio of order N, such as how we related φ to a regular pentagon?

There is more to explore beyond the scope of this worksheet. For example, in addition to the metallic ratios above, there is also a quantity called *the plastic ratio*. This ratio can be defined as the unique positive root ρ (the Greek letter rho) to the *cubic* equation

$$\rho^3 = \rho + 1.$$

Rather than arising from rectangles and squared, the plastic ratio ρ first arises from equilateral triangles. The plastic ratio also arises in the context of the *Padovan Sequence*, another sequence defined in ways analogous to how the Fibonacci and Pell sequences are defined. Namely, the Padovan Sequence (P_n) is defined by The *Pell Sequence* is the sequence $(P_n) = P_1, P_2, P_3, \ldots$ defined recursively via

$$P_1 := 1, \tag{4.10}$$
$$P_2 := 1, \tag{4.11}$$
$$P_3 := 1, \text{ and} \tag{4.12}$$
$$P_{n+3} := P_{n+1} + P_n \text{ for all } n \geq 1. \tag{4.13}$$

In other words, rather than summing the immediate two preceding terms to get the next one, skip the immediate predecessor, and sum the two terms before that one.

The Padovan Sequence begins

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, \ldots,$$

because $2 = 1 + 1$, $2 = 1 + 1$, $3 = 2 + 1$, $4 = 2 + 2$, $5 = 3 + 2$, $7 = 4 + 3$, $9 = 5 + 4$, $12 = 7 + 5$, etc.

For more on the plastic ratio, see [5].

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