Chapel Hill Math Circle

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Session 8 – January 18, 2025:
The Frobenius Coin Problem:
What Amounts you can pay with Coins
Beginners' Group (grades 1-3), 10:30-11:30a
Mr. Barman – dilip@trianglemathinstitute.com

Supplies needed per student: 10 each of 3 different coins or tokens such as pennies, nickels, and dimes, as well as graph paper and pencil

Welcome to Math Circle session 8 and Happy New Year! Let's look at paying with coins. Can we make any amount with a set of coins?

Let's start easy and work with pennies

If you had an unlimited number of pennies, what amounts could you pay? This is meant to be a trivial question.

How are things different if you have nickels and dimes only? Or nickels, dimes, quarters, and dollar bills? What is special about pennies?

Let's use two coins one worth 3¢ and the other 4¢ - what amounts can you make?

Clearly you can make 3¢, 6¢, 9¢, 12¢, and any "multiple" of 3 cents.
You can also make 4¢, 8¢, (12¢), and any multiple of 4 cents.
But you can also make (3+4= 7)¢, (3+3+4 = 13)¢, and a whole lot of other amounts. Take a few minutes and see if you can figure out what amounts you can make – or, if you find it easier, find what amounts you can't make. You can't make 1¢, 2¢, or 5¢, right? You can use two of your physical coins that you have – maybe a penny is worth 3¢ and a nickel is worth 4¢.

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You should have found that you can actually make any amount of whole cents *except* for 1, 2, and 5 cents. Can you show why this is true? Let's look at a more difficult problem to see how we can show this in general.

What about 3¢ and 11¢

Take a few minutes and see what amounts can and can't be made with 3 and 11 cents. Remember you have these two coins and in theory have an unlimited number of each coin. Don't worry if you can't figure this out but give it a try.

Let's take a detour to skip-counting

You're familiar with skip-counting of course. You probably have built a tens chart to skip count by 10, maybe looking something like this:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160

With a tens chart like this we can start at, say, 27, and skip count by 10 by going straight down.

You could also make a fives chart to skip count by 5 or a 123s chart to skip count by 123, for example. But … what does this have to do with our problem? Any ideas?

Here's the secret … if I make not a tens chart but an 11s chart I will be on the way to finding a solution. Let's explore this together. Take notes and see if you find that we can make any amount of whole cents except for 1, 2, 4, 5, 7, 8, 10, 13, 16, and 19 cents. In particular, **19 cents is the largest amount that you can't make with 3 and 11 cent coins; you can make any larger amount**!

Take graph paper or use my 11-across grid below[1] and prove this to yourself by making an 11s chart (or a 3s chart). Once you circle an amount that you can make, cross it out and all numbers directly below it (why?); these are numbers that can be made. Try it out!

More to think about

Can you easily look at two numbers and decide if:

- they can make all amounts (like any case where there's a penny)
- they obviously can't make all amounts (e.g., with a nickel and a dime)
- they can make almost all amounts and you can find the largest amount that can't be made

What about if you consider three coins? More? Does the parity (whether the number is even or odd) matter? Have fun exploring the Frobenius Coin Problem!

Have Fun!
Mr. Barman

P.S. If you know multiplication, did you know that 2025 is a perfect square? What number times itself gives you 2025? What is the next perfect square year?

[1] I easily made the grid on the website incompetech.com/graphpaper/lite Jan. 17, 2025.

Notes for Parents

The Frobenius Coin Problem is named after Ferdinand Georg Frobenius[2] (1849-1917). An example problem is the one that we are working out today. Imagine that you have an unlimited supply of 3¢ and 11¢ coins. With these coins you can make amounts like 3¢, 6¢, 11¢, 22¢, (3+11=14) 14¢, ((5x10)+(10x11) = 160) 160¢, and infinitely many more. What is the largest amount that you cannot make? Is there such a largest amount?

Clearly you can make an infinite number of amounts; trivially, you can use any number of either coin so, for example, you can make any multiple of 3¢. There are an infinite number of multiples of 3.

If we build a table like the one on the next page, we get our answer of 19¢ as being the largest amount we can't make. In fact the table tells us additionally that:

- We can't make 1¢, 2¢, 4¢, 5¢, 7¢, 8¢, 10¢, 13¢, or 16¢, but we can make any other (positive integer) number of cents!

- We can easily figure out how to make these sums. For example, consider 53¢. 53 is under 9 (which is three 3s); in particular, it is four rows (of 11) below 9. So 53 = (3x3) + (4x11) and 53¢ can therefore be made by three 3¢ coins and four 11¢ ones.

That's the "trick" – we make a skip counting table! (I submitted my handout late and apologize that this is probably printed in black and white; the table uses a different color for each step described with 11, 3, 6, etc. I hope that it still makes sense monochromatically.

[2] Picture from wikipedia.org/wiki/Ferdinand_Georg_Frobenius, accessed Oct. 7, 2024, and freely usable

1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22
23	24	25	26	27	28	29	30	31	32	33
34	35	36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63	64	65	66
67	68	69	70	71	72	73	74	75	76	77
78	79	80	81	82	83	84	85	86	87	88
89	90	91	92	93	94	95	96	97	98	99
100	101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120	121
122	123	124	125	126	127	128	129	130	131	132
133	134	135	136	137	138	139	140	141	142	143
144	145	146	147	148	149	150	151	152	153	154
155	156	157	158	159	160	161	162	163	164	165
166	167	168	169	170	171	172	173	174	175	176

Can make 11 and everything under it (11, 11+11, 11+11+11, etc.)

Can make 3 and everything under it (3, 3+11, 3+11+11, etc.)

Can make 6 and everything under it

Can make 9 and everything under it

Can make 12 and everything under it

Can make 15 and everything under it

Can make 18 and everything under it

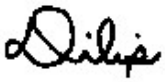
Can make 21 and everything under it

So the largest number we can't make with 3s and 11s is 19

The Frobenius Number of 3 and 11 is 19; we say g(3, 11) = 19

I hope that this has been an interesting and unique problem for your children to consider. I cover it in my level 3 and 4 classes at Mathematics Institute of the Triangle; it was even a test question!

Till next time!



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P.S. As to the 2025 question, note that 40^2 is 1600 and 50^2 is 2500. If 2025 is a perfect square its square root must end in 5 and it must be between 40 and 50 – it must be 45. Do you know the trick that easily shows us that 45^2 = (40x50)+ 25 = 2025? (A diagram will show this.) The next perfect square year is $46^2 = 45^2 + 45 + 46$ = 2025 + 91 = 2125 – 9 = 2116. Your children will be around 100 years old then!