CHAPEL HILL MATH CIRCLE EXIT TICKET:
March 15, 2025: Polynomials

Please remove this sheet, complete it, and return it before the end of our session.

- Did you find today's topic interesting?

- Was the this topic appropriately challenging relative to your background? That is, was the topic neither too elementary nor inaccessibly advanced?

- How could we improve this worksheet for future sessions?

- What did you enjoy about today's topic?

- What did you find particularly challenging?

- Was there anything you thought was too difficult?

- Was there anything you thought was too easy?

- Are there any topics you would be interested in seeing us cover in the future?

Vieta's Formulas and Symmetric Polynomials

Abstract

This week, we shall explore properties of polynomials, beginning with polynomials in one variable of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer, and $a_0, a_1, \ldots, a_n$ are coefficients. In particular, we consider properties of the set of *roots* (or *zeroes*) of such polynomials.

Using *Vieta's*[1] *Formulas* as motivation, we then consider multivariable polynomials and *symmetric polynomials* in particular. Building from some these results, we shall explore connections between symmetric polynomials and the roots of a polynomial in one variable.

Background needed: Prerequisites include familiarity with algebra, especially factoring polynomials and determining their roots. Awareness of complex numbers and their properties will also be useful.

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[1] Named for *François Viète*, or Franciscus Vieta in Latin. Viète (1540–1603) was a French lawyer and mathematician.

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0 Warmup

As prerequisites for this session, it will help to answer the following first. You do *not* need to know these answers already, and many questions will be revisited later in the worksheet.

0.1 What is a *polynomial* in the single variable x? What is a polynomial in multiple variables, such as x, y, z? Give some examples of both single-variable and multivariable polynomials.

0.2 Let $p(x)$ be a polynomial with coefficients in the complex numbers, denoted $\mathbb{C}$. What is a *root* (or *zero*) of $p(x)$? What is the *multiplicity* of a root of a polynomial?

1 Polynomials in One Variable

1.1 Discussion

Definition 1.1.1. The *polynomials* in x with complex coefficients, denoted $\mathbb{C}[x]$, is the set

$$\left\{p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 : n \in \mathbb{Z}, n \geq 0, a_0, a_1, \ldots, a_n \in \mathbb{C}\right\}. \tag{1.1.1}$$

That is, $p(x)$ is a polynomial in x if it is the sum over finitely many terms of the form $a_k x^k$, where each a_k is a complex number and each k is a nonnegative integer.

Addition and multiplication of polynomials are defined in the usual way.

We can also consider polynomials with different coefficients, such as over the integers (itself denoted $\mathbb{Z}$, with the set of polynomials denoted $\mathbb{Z}[x]$), over the rational numbers (denoted $\mathbb{Q}$ and $\mathbb{Q}[x]$, respectively), or over the real numbers (denoted $\mathbb{R}$ and $\mathbb{R}[x]$, respectively). We can also consider coefficients in, for example, $\mathbb{Z}/m\mathbb{Z}$, the integers modulo m. In what follows, though, we shall limit our attention to polynomials with complex coefficients and their complex roots, though many of the results will generalize beyond $\mathbb{C}$.

Notation. Where the context is clear, we shall use notation like $p(x)$ and p interchangeably.

Example 1.1.2. Each of the following is an example of a polynomial with complex coefficients:

- 4, the constant polynomial
- $5x^2 + 2x - 3$
- $-\frac{17}{3}x^4 - 2x + \frac{81}{16}$
- $-\pi x^5 + 13x^3 - e^{\sqrt{2}}x + \frac{5}{21}$
- $(1-2i)x^4 + \left(\frac{7}{22} + (\log 8)i\right)x^3 - 14x^2 + \frac{87}{13}ix + \left(\cos\frac{2\pi}{7} + \sin\frac{2\pi}{7}i\right)$

Definition 1.1.3. Let $p(x)$ be a polynomial. The complex number r is a *root* (or *zero*) of p if and only if $p(r) = 0$.

That is, a root of a polynomial is a complex number r such that when you evaluate p at r, the result is zero.

1.2 Exercises

1.2.1 What is the *degree* of a polynomial p, denoted $\deg p$? (Ideally, you should be able to answer this for polynomials in one variable, as well as in multiple variables.)

1.2.2 First, we explore a relationship between degree 1 factors of polynomials and their roots:

Theorem 1.2.1 (Polynomial Remainder Theorem)**.** *Let $p \in \mathbb{C}[x]$ and $c \in \mathbb{C}$. Then the remainder when dividing $p(x)$ by $x - c$ is the constant $p(c)$. (That is, p is expressible in the form $p(x) = (x-c)q(x) + p(c)$ for some polynomial q, and where $p(c)$ is the constant.)*

Note: Our priority is that you understand and can use this theorem later. Being able to prove it would be a bonus, but secondary.

1.2.3 Next, we present an important corollary to Theorem 1.2.1:

Corollary 1.2.1(a). *If $p \in \mathbb{C}[x]$ and $c \in \mathbb{C}$, then $x - c$ divides p if and only if $p(c) = 0$.*

Note: Again, the priority is being able to understand and apply this Corollary, not prove it.

1.2.4 How would you define polynomials in multiple variables? For example, if x and y are indeterminates, how might you define $\mathbb{C}[x,y]$, the set of polynomials over $\mathbb{C}$ in both x and y? What about $\mathbb{C}[x_1,x_2,\ldots,x_n]$?

2 The Fundamental Theorem of Algebra and Vieta's Formulas

2.1 Discussion

We begin with the following, whose proof is *far* beyond the scope of this session:

Theorem 2.1.1 (The Fundamental Theorem of Algebra)**.** *Let $p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with coefficients $a_0, a_1, \ldots, a_n$ lying in $\mathbb{C}$, and $a_n \neq 0$. If p is not a constant polynomial, then p has at least one root in $\mathbb{C}$.*

In other words, any polynomial with complex coefficients, except for constant polynomials, must have at least one complex root.

Remark. To better appreciate the value of The Fundamental Theorem of Algebra, we note that it uses *in an essential way* that the coefficients and roots of our polynomial both lie in $\mathbb{C}$, the field of complex numbers. For example:

- The polynomial $3x-2 \in \mathbb{Z}[x]$ has the unique root $r := \frac{2}{3}$, but this root is not *an integer*.
- The polynomial $x^2-2 \in \mathbb{Q}[x]$ has the roots $\pm\sqrt{2}$, but these are not *rational* numbers.
- The polynomial $x^2+1 \in \mathbb{R}[x]$ has the roots $\pm i$, but these are not *real* numbers.

Corollary 2.1.1(a). *Let $p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $n \geq 1$. Then there exist $r_1, r_2, \ldots, r_n \in \mathbb{C}$, not necessarily distinct, such that*

$$p(x) = a_n (x-r_1)(x-r_2)\cdots(x-r_n).$$

That is, a nonconstant complex polynomial of degree n has precisely n complex roots, including multiplicity.[2]

[2]The *multiplicity* of a root r_j of the nonzero polynomial p is the largest positive integer n_j such that $(x-r_j)^{n_j}$ divides p. This corollary therefore counts not just how many distinct roots p has, but also their multiplicity.

The Fundamental Theorem and its corollary tell us that every nonconstant complex polynomial "splits linearly", meaning it is expressible as product of one nonzero constant and n monic[3] polynomials of degree 1. Now that we know every nonconstant polynomial over $\mathbb{C}$, let us use this to explore the relationship between the roots and coefficients of a polynomial in the exercises.

Example 2.1.2. Let a, b, c be real (and therefore complex) numbers such that $a \neq 0$. Then the quadratic polynomial

$$p(x) = ax^2 + bx + c \tag{2.1.1}$$

has its two roots r_1, r_2 given by the quadratic formula:

$$r_i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.1.2}$$

If the discriminant $b^2 - 4ac = 0$, then $r_1 = r_2$ is a repeated root of multiplicity 2. Otherwise, the two roots are distinct. If $b^2 - 4ac > 0$, these roots are real. If $b^2 - 4ac < 0$, the roots are nonreal complex conjugates of each other.

2.2 Exercises

2.2.1 Let $p(x) := x^2 + 4x + 10$, and let r, s denote its respective roots. *Note:* r and s are nonreal complex numbers.

Determine a quadratic polynomial $q(x)$ such that the roots of q are precisely r^2 and s^2. Can you compute $q(x)$ *without* first computing the values r and s?

2.2.2 Consider the polynomial $p(x) := 2x^2 + 3x - 5$. By The Fundamental Theorem of Algebra, p has precisely two roots, which we shall denote by r and s. Compute $r + s$ and rs. Can you do so *without* first computing r and s?

2.2.3 Let $p(x) := 5x^3 - 14x^2 - 2x + 8$, and denote its roots (including possible repetitions) by r_1, r_2, r_3. Compute the values

$$r_1 + r_2 + r_3$$
$$r_1 r_2 + r_1 r_3 + r_2 r_3$$
$$r_1 r_2 r_3.$$

[3] If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial, we say p is *monic* if and only if $a_n = 1$, where $n := \deg p$.

2.2.4 Let $p(x) := 3x^5 + 17x^4 - 12x^3 - 68x^2 + 12x + 68$, and denote its roots (including possible repetitions) by r_1, r_2, r_3, r_4, r_5. Compute the values

$$r_1 + r_2 + r_3 + r_4 + r_5 \text{ and } r_1 r_2 r_3 r_4 r_5.$$

In the context of Exercise#2.2.3, what other expressions in the r_j can we also compute?

2.2.5 Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial over $\mathbb{C}$ with degree $n \geq 1$. Further, let $r_1, r_2, \ldots, r_n$ be the roots of p, including possible repetitions. Compute the values

$$\begin{gathered} r_1 + r_2 + \cdots + r_n \\ r_1 r_2 + \cdots r_1 r_n + r_2 r_3 + \cdots r_2 r_n + \cdots + r_{n-1} r_n \\ r_1 r_2 r_3 + \cdots + r_{n-2} r_{n-1} r_n \\ \vdots \\ r_1 r_2 \cdots r_{n-1} + \cdots + r_2 r_3 \cdots r_n \\ r_1 r_2 \cdots r_n \end{gathered}$$

in terms of the coefficients $a_0, a_1, \ldots, a_n$. The resulting equations are called *Vieta's Formulas*.

2.2.6 Let r, s be the roots of the polynomial $p(x) := x^2 + 4x + 7$.

What is the value of

$$r^3 + s^3?$$

Can you compute the value of

$$r^2 s + r s^2 + 11 r s,$$

as well?

3 Additional Exercises with Vieta's Formulas

We now consider some more creative applications of Vieta's Formulas, many of which have been taken from mathematics competitions.

3.1 Let $p(x) := x^3 + 2x - 4$ have roots r_1, r_2, r_3. Prove that for each j, $r_j^3 = -2r_j + 4$. Using this, compute the sum $r_1^3 + r_2^3 + r_3^3$.

More generally, let $f(x) := a_n x^n + a_1 x + a_0$, where n is a positive integer, a_n, a_1, and a_0 are complex numbers, and $a_n \neq 0$. Let $f(x)$ have roots $r_1, r_2, \ldots, r_n$. Use the idea above to simplify computation of the sum

$$r_1^n + r_2^n + \cdots + r_n^n. \tag{3.0.1}$$

Note: The first example is taken from [7], who calls this the *Power Reduction Trick*.

3.2 The roots of $x^2 - 6x + 10$ are r_1 and r_2. Find a polynomial of degree 2 whose roots are precisely $\frac{r_1}{r_2}$ and $\frac{r_2}{r_1}$.

Note: This exercise is slightly modified from [8].

3.3 The following is taken from [4]: "Find all complex numbers a, b, c for which $a + b + c = 0$, $ab + ac + bc = 0$, $abc = 0$."

3.4 **American Invitational Mathematics Examination (AIME), Problem #5:** "Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b$, $b + c$, and $c + a$. Find t."

3.5 **American Mathematics Competitions' AMC 12, Problem #19:** "The polynomial $p(x) = x^3 + ax^2 + bx + c$ has the property that the average of its zeros, the product of its zeros, and the sum of its coefficients are all equal. The y-intercept of the graph of $y = p(x)$ is 2. What is b?

(A) -11 (B) -10 (C) -9 (D) 1 (E) 5"

4 Playing with Polynomials

4.1 Let x be a number such that

$$x + \frac{1}{x} = 1.$$

What is the value of

$$x^2 + \frac{1}{x^2}?$$

Can you compute the value of

$$x^3 + \frac{1}{x^3}$$

as well? Can you further generalize?

4.2 Let $\alpha \in \mathbb{C}$. We say that α is an *algebraic number* if and only if there exists some nonconstant polynomial $p \in \mathbb{Q}[x]$ such that $p(\alpha) = 0$. One can show that for every algebraic number, there exists a unique *minimal polynomial* $p \in \mathbb{Q}[x]$ such that (a) $p(\alpha) = 0$, (b) p is irreducible in $\mathbb{Q}[x]$, and (c) p has n *distinct* roots in $\mathbb{C}$, where $\deg p = n$.

Assume that α and β are algebraic numbers. Prove that $\alpha + \beta$ and $\alpha\beta$ are also algebraic numbers.

Hint: Say α and β have minimal polynomials p and q, respectively, where $n := \deg p$ and $m := \deg q$. Consider the *conjugates* $\{\alpha_i\}$ of α, the collection of all n complex roots of p, and $\{\beta_j\}$, the set of all m complex roots of q. Define polynomials

$$S(x) := \prod_{1 \le i \le n} \prod_{1 \le j \le m} \big(x - (\alpha_i + \beta_j)\big) \tag{4.0.1}$$

and

$$P(x) := \prod_{1 \le i \le n} \prod_{1 \le j \le m} \big(x - (\alpha_i \beta_j)\big). \tag{4.0.2}$$

What can we say about the coefficients of S and P?

4.3 Let $p \in \mathbb{Q}[x]$ be a monic polynomial of degree $n \ge 1$ such that the roots of p, including multiplicity, are $r_1, r_2, \ldots, r_n$. Define the *discriminant of p* to be the number

$$\operatorname{Disc} p := \prod_{1 \le i < j \le n} (r_i - r_j)^2.$$

- Prove that if $p \in \mathbb{Q}[x]$, then $\operatorname{Disc} p$ is a *rational* number.

- Consider $p(x) := x^2 + b + c$. Compute Disc p.
- Show that in general, Disc $p = 0$ if and only if p has a repeated root.

5 Multivariable Polynomials and Symmetric Polynomials

5.1 Discussion

Vieta's Formulas help us connect the coefficients of a polynomial p in one variable to the relevant expressions in the roots $r_1, r_2, ..., r_n$ of p. Note that these expressions are multivariable polynomial expressions in the r_j, too. For example, if $e_n(x_1, x_2, \ldots, x_n) := x_1 x_2 \cdots x_n$, then evaluating at the point $(r_1, r_2, \cdots, r_n) \in \mathbb{C}^n$, we have

$$e_n(r_1, r_2, \cdots, r_n) = (-1)^n \cdot \frac{a_0}{a_n}.$$

Further, e_1 is such that any permutation of the variables $x_1, x_2, \ldots, x_n$ does not change the polynomial e_1. (For example, $e_1(x_1, x_2, x_3) = e_1(x_3, x_1, x_2) = e_1(x_2, x_1, x_3)$, etc.) The form of the identites in Vieta's Formulas therefore motivates us to properties of certain classes of multivariable polynomial functions.

Definition 5.1.1. Let $p(x_1, x_2, \ldots, x_n) \in \mathbb{C}[x_1, x_2, \ldots, x_n]$. Then p is a *symmetric polynomial* if and only if for every permutation[4] τ on the set $\{1, 2, \ldots, n\}$,

$$p(x_1, x_2, \ldots, x_n) = p\left(x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(n)}\right).$$

Example 5.1.2. The following are examples—and nonexamples—of symmetric polynomials in $\mathbb{C}[x_1, x_2, \ldots, x_n]$:

- $p(x, y) := x^2 + 5xy + y^2$ is symmetric in $\mathbb{C}[x, y]$
- $p(x, y) := x^2 + 5xy - 2y^2$ is not symmetric in $\mathbb{C}[x, y]$

  To see this, note that $p(y, x) = y^2 + 5xy - 2x^2$, and therefore $p(x, y) \neq p(y, x)$. Therefore, the permutation that transposes x and y shows that p is not symmetric.
- For every nonnegative integer k,

  $$\sigma_k(x_1, x_2, \ldots, x_n) := x_1^k + x_2^k + \cdots + x_n^k$$

  is symmetric in $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Each σ_k is called the *power sum polynomial of degree k*.

[4] *Question:* Do you understand what a permutation on a set S is? If not, please ask!

- If $a \in \mathbb{C}$ is a constant, and p, q are symmetric polynomials in $\mathbb{C}[x_1, x_2, \ldots, x_n]$, then so are ap, $p+q$, and pq.
- Whether a polynomial is symmetric depends not just on the polynomial, but on the ambient space of polynomials.

  For example, $x^2+5xy+y^2$ is symmetric in $\mathbb{C}[x,y]$, but it is *not* symmetric in $\mathbb{C}[x,y,z]$. In the latter ring, $p(x,z,y) = x^2+5xz+z^2 \neq x^2+5xy+y^2 = p(x,y,z)$.

Definition 5.1.3. The *elementary symmetric polynomials* in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ are the following:

$$\begin{aligned}
e_0(x_1, x_2, \ldots, x_n) &:= 1 \\
e_1(x_1, x_2, \ldots, x_n) &:= x_1 + x_2 + \cdots + x_n \\
e_2(x_1, x_2, \ldots, x_n) &:= \sum_{1 \le j_1 < j_2 \le n} x_{j_1} x_{i_2} \\
e_3(x_1, x_2, \ldots, x_n) &:= \sum_{1 \le j_1 < j_2 < j_3 \le n} x_{j_1} x_{j_2} x_{j_3} \\
&\vdots \qquad\quad \vdots \\
e_k(x_1, x_2, \cdots x_k) &:= \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} x_{j_1} x_{j_2} \cdots x_{j_k} \\
&\vdots \qquad\quad \vdots \\
e_{n-1}(x_1, x_2, \ldots, x_n) &:= \sum_{1 \le j_1 < j_2 < \cdots < j_{n-1} \le n} x_{j_1} x_{j_2} \cdots x_{j_{n-1}} \\
e_n(x_1, x_2, \ldots, x_n) &:= x_1 x_2 \cdots x_n.
\end{aligned}$$

That is, for each k with $1 \le k \le n$, each e_k is the sum over all distinct k-at-a-time products over $\{x_1, x_2, \ldots, x_n\}$.

Example 5.1.4.

- In $\mathbb{C}[x,y,z]$, we have

$$\begin{aligned}
e_1(x,y,z) &:= x+y+z \\
e_2(x,y,z) &:= xy+xz+yz \\
e_3(x,y,z) &:= xyz.
\end{aligned}$$

- For every positive integer $n \ge 2$, in $\mathbb{C}[x_1, x_2, \ldots, x_n]$, we have

$$\begin{aligned}
e_{n-1}(x_1, x_2, \ldots, x_n) := {} & x_1 x_2 \cdots x_{n-2} x_{n-1} \\
& + x_1 x_2 \cdots x_{n-2} x_n \\
& + x_1 x_2 \cdots x_{n-3} x_{n-1} x_n \\
& + \cdots \\
& + x_1 x_3 \cdots x_{n-1} x_n.
\end{aligned}$$

- Let $p(x) \in \mathbb{C}[x]$ be a polynomial of degree n, and whose roots (including multiplicity) are $r_1, r_2, \ldots, r_n$. Then we can express Vieta's Formulas (Exercise #2.2.5) in terms of elementary symmetric polynomials:

$$\begin{aligned} e_1(r_1, r_2, \ldots, r_n) &= -\frac{a_{n-1}}{a_n} \\ e_2(r_1, r_2, \ldots, r_n) &= \frac{a_{n-2}}{a_n} \\ \vdots &= \quad \vdots \\ e_k(r_1, r_2, \ldots, r_n) &= (-1)^k \cdot \frac{a_{n-k}}{a_n} \\ \vdots &= \quad \vdots \\ e_n(r_1, r_2, \ldots, r_n) &= (-1)^n \cdot \frac{a_0}{a_n}. \end{aligned}$$

5.2 Exercises

5.2.1 Consider the symmetric polynomial $p(x, y) := x^3 + y^3 \in \mathbb{C}[x, y]$. Express p in terms of the elementary symmetric functions in $\mathbb{C}[x, y]$.

5.2.2 Consider the symmetric polynomial $p(x, y, z) := x^2 + y^2 + z^2 \in \mathbb{C}[x, y, z]$. Express p in terms of elementary symmetric functions.

5.2.3 Let $p(x, y, z) \in \mathbb{C}[z, y, z]$ be a symmetric polynomial containing the monomial xyz^2. What other terms must appear in p?

5.2.4 **Challenging:** Prove the following theorem.

Theorem 5.2.1 (The Fundamental Theorem of Symmetric Polynomials)**.** *Let p be any symmetric polynomial in $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Prove that there exists some polynomial q—itself not necessarily symmetric!—such that $q \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ and*

$$p(x_1, x_2, \ldots, x_n) = q\left(e_1(x_1, x_2, \ldots, x_n), e_2(x_1, x_2, \ldots, x_n), \ldots, e_n(x_1, x_2, \ldots, x_n)\right).$$

Furthermore, q is uniquely determined by by p.

References

[1] Art of Problem Solving. Art of Problem Solving: Tackling problems with Vieta. https://www.youtube.com/watch?v=0QHnVMWQNfI, August 3, 2012.

[2] David S. Dummit and Richard M. Foote. *Abstract Algebra*. John Wiley and Sons, Inc., third edition, 2004.

[3] Chan Lye Lee. Vieta's formula: a simple example. https://www.youtube.com/watch?v=IuvaJvSXh3I, July 31, 2021.

[4] Mohamed Omar. Vieta's formula. https://www.youtube.com/watch?v=OWtXsA2CAOA, September 13, 2021.

[5] Sohil Rathi. Vieta's Formulas - mastering AMC 10/12. https://www.youtube.com/watch?v=czk6OsfrbxQ, October 24, 2022.

[6] Sohil Rathi. Polynomial root manipulations - mastering AMC 10/12. https://www.youtube.com/watch?v=Dp-pw6NNKRo, October 25, 2022.

[7] RecurringRoot. Vieta's formulas + example problems. https://www.youtube.com/watch?v=uxjqKuT6Lx4, August 18, 2019.

[8] SyberMath. Vieta's formulas with examples. https://www.youtube.com/watch?v=zx6Grk_aJNs, December 7, 2020.

[9] SyberMath. Vieta's formulas in a cubic equation. https://www.youtube.com/watch?v=3nAZ6vJIpLg, January 11, 2021.