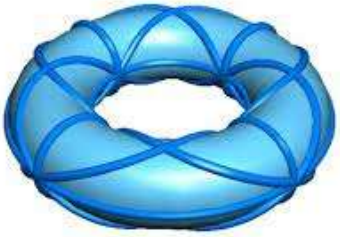
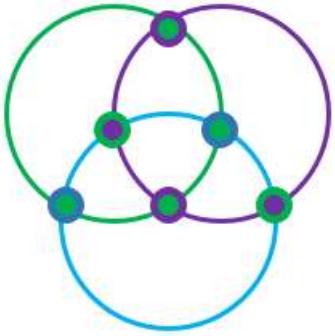


Chapel Hill Math Circle

Chapel Hill Math Circle
Session 13 – March 29, 2025:
Handshakes and Intersecting Circles
Beginners' Group (grades 1-3), 10:30-11:30a
Mr. Barman – dilip@trianglemathinstitute.com



Supplies needed per student: 5-6 rings of any convenient diameter, perhaps between 1 and 3 inches (paper rings, bangles, or even rubber bands), as well as paper and pencil

Welcome to Math Circle session 13. We will explore the idea of bounds, counting handshakes, and counting the number of times circles might intersect.

Upper and lower bounds

Sometimes we don't know the answer to a question because different things can happen or because we just don't know. Let's talk about bounds.

Let's try to answer the question how many times a 2nd grader sits down to eat pizza in a week. We don't know much about the 2nd grader except her name (Sylvia). We don't know her or know anything about her. What would you guess would be the number of times in a given week that Sylvia sits down to eat pizza?

There's no way to know how many times she'll sit down to eat pizza on any given week, but we can **bound** the answer. The lower bound is zero. It's certainly possible she'll not eat any pizza on a specific week. There are people in the world who have never eaten pizza and who don't even know what it is.

What's the upper bound?

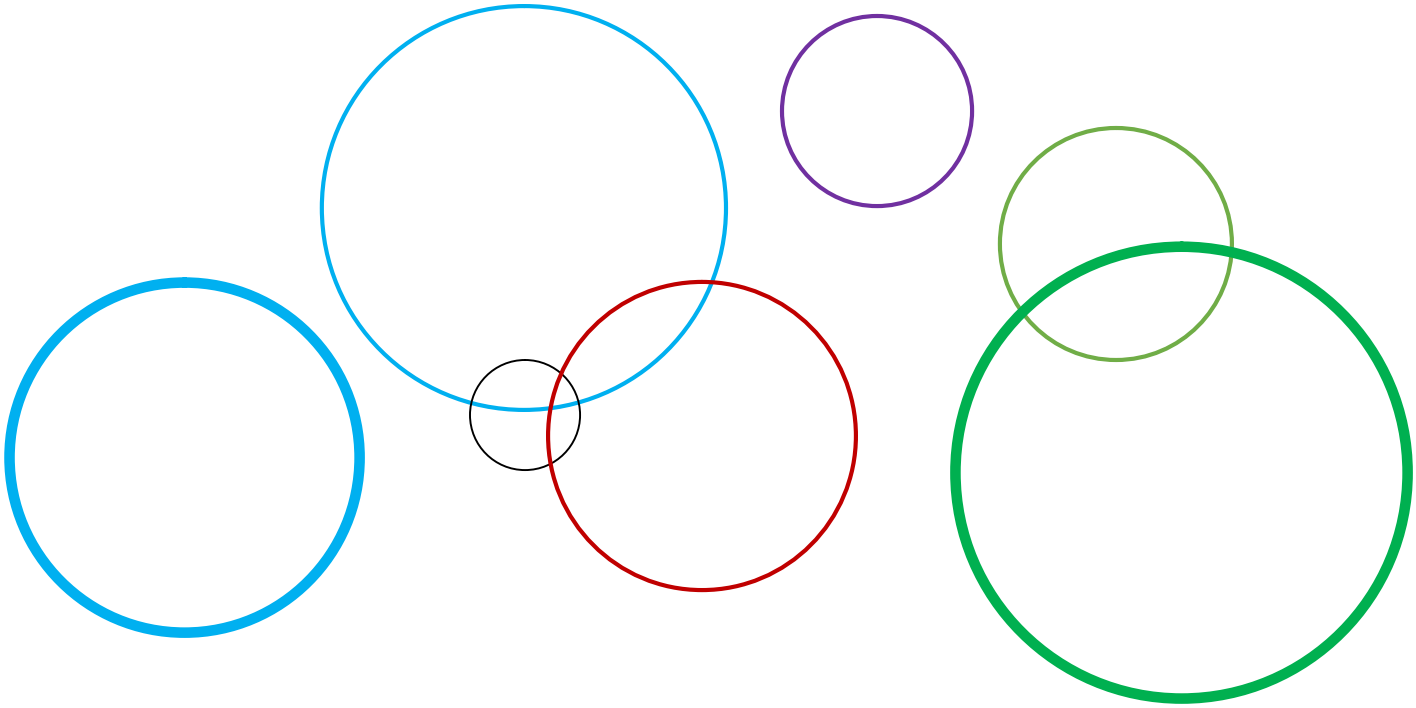
- Sylvia may be a pizza lover and may eat pizza at every meal. There are typically 3 meals a day and 7 days so (3+3+3+3+3+3+3 = 7+7+7 =) 21 meals per week. Is that our upper bound?
- Actually, Sylvia may have an afternoon snack of pizza every day. Is our upper bound now 21+7 = 28?
- What if Sylvia is a real pizza lover and eats pizza every hour – even waking up at night to eat pizza. There are 24 hours in the day and 7 days a week. That's something like 7 25s (I'd rather skip count and overestimate a bit – it's easy to think about quarters) or 175. Is that our upper bound?
- What if she loves pizza so much that for one week she gives up everything – sleep, going to school, playing, and constantly eats pizza. Here it would be 1 time as she would sit down and eat for a week. But if we assume that she gets up every minute it will be a large number.
- What if she eats constantly but gets up twice a minute? Ten times a minute? This would basically mean there's no upper bound.

Math is great but we need to always be informed by common sense. We have argued that the lower bound of times Sylvia will sit down to eat pizza is 0 and there is no upper bound. But we can make practical bounds.

- The lower bound of 0 still holds.
- Realistically, would somebody eat pizza 3 meals a day for 7 days? I don't think so. It's possible but highly unlikely. A more reasonable upper bound might be 7 – Sylvia might eat pizza every day but just once. Again we can't be sure but I think this stretches what is likely to be true.
- How about a likely upper bound? Most people who love a food wouldn't eat it every single day. A likely upper bound might be 3 or 4 times a week?

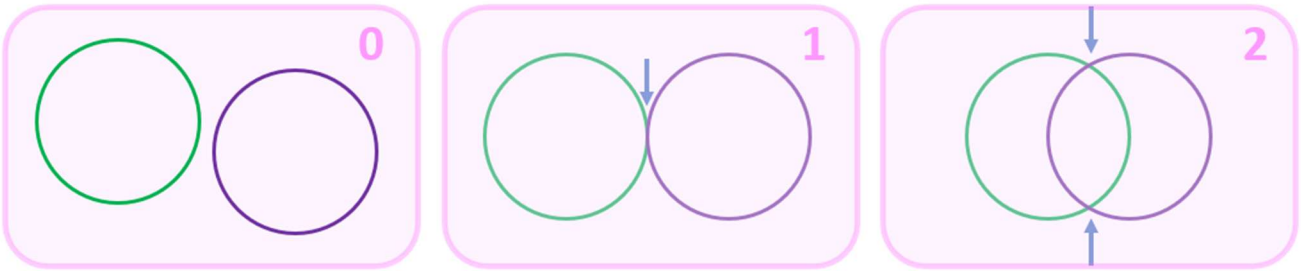
Can you determine how many times two circles can intersect each other?

Can you find a lower bound and an upper bound? Should we specify that the two circles aren't right on top of each other – i.e., that they are *non-overlapping*? In this case they would overlap an infinite number of times. Use your rings or rubber bands to experiment.

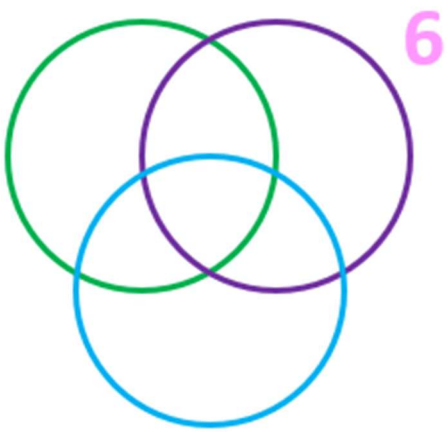


Let's assume that the two circles are not overlapping – if they are overlapping, then they intersect at an infinite number of places. Does size matter?

If we have two circles, they can intersect nowhere, at one point, or at most at two points (remember that we are ignoring the case of putting circles atop each other where there would be an infinite number of intersections). So the upper bound of points of overlap is 2.



How about three circles?



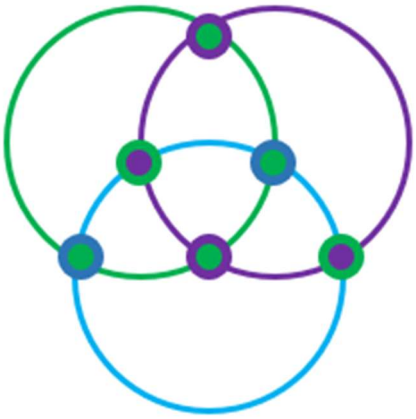
We can see that they can touch each other in at most 6 points. Is that what you found as well?

Let's prove that we are correct.

- Let's think about the top left green circle. It can touch the top right purple one twice and can touch the bottom blue one twice.
- Now let's think about the top right purple one. We have already counted the touches with the green circle – but we still have to count the two times it can touch the bottom blue one.
- The blue one has nothing more to count!

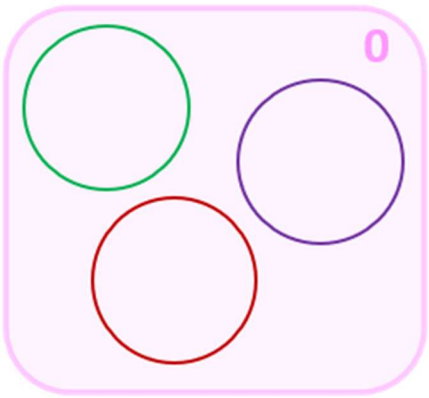
So we have (2+2) + (2) = 6 touch points. Let's make sure that we understand this by making a table and another picture.

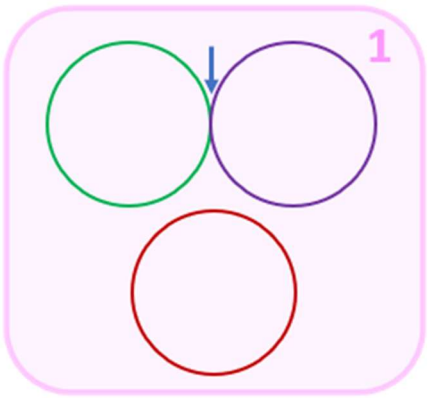
	Intersections with Purple	Intersections with Blue	Total Intersections
Green	2	2	2+2 = 4
Purple		2	2
Blue			0

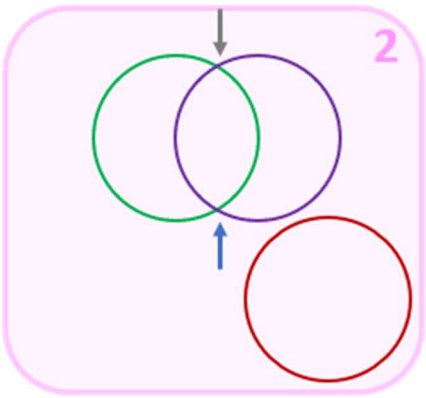


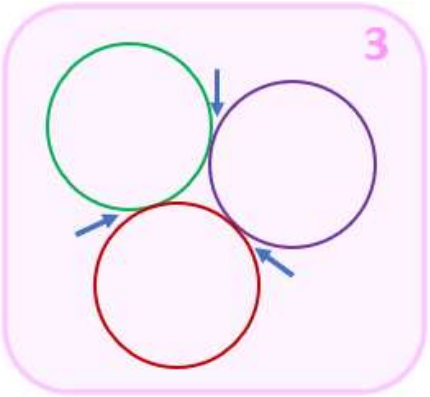
I can show that you can get 0, 1, 2, 3, 4, 5, or 6 intersections. Here are 0-2 below. We just moved the purple circle to the left to make it intersect the green one one more time.

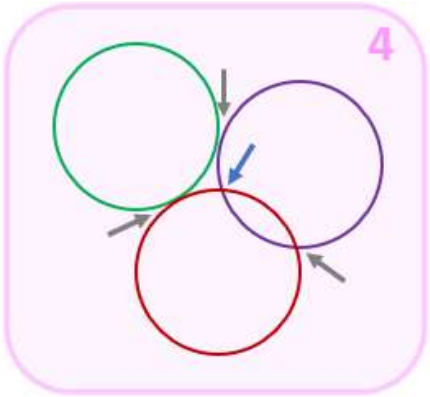
Now I move all circles to just kiss each other to get 3 intersections. I move red (really maroon) to get a second touch with purple and then move purple to get a second touch with green.

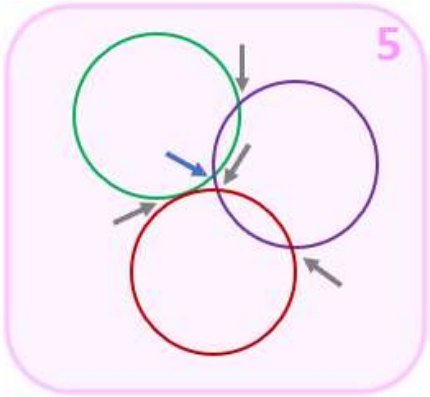




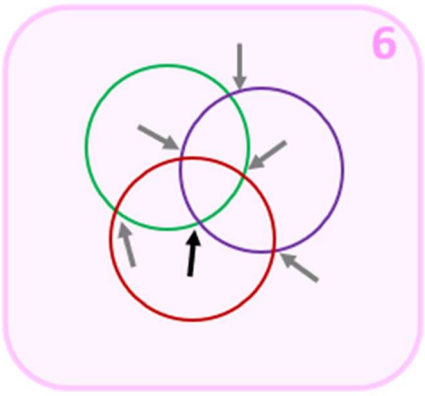






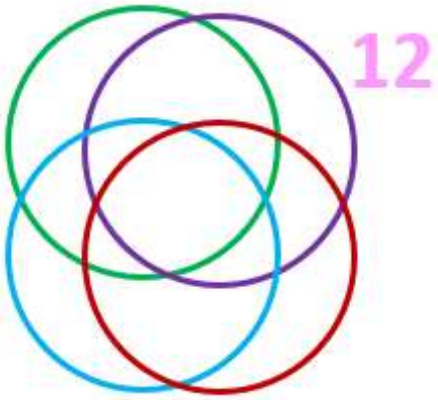


Finally, here is 6. I move red in so it gets a second touch with green; each circle is now connected two places to every other circle – from red's perspective we have 2 touches with purple and 2 with green; from purple's perspective we've counted the touches with red but we have 2 with green. From green's perspective everything's counted so we have 0 more – (2+2) + (2) + (0) = 6.



When we find an upper bound, say 12 intersections for 4 circles, can we always get every number up to it? Think about this at home. When you answer that if you know for sure, give proof; if you don't know, share your intuition – what you think is true – and why.

How about four circles?



The upper bound for four circles is 12. Is that what you got?

Let's prove that we are correct. Let's call the top left green circle A, the top right purple circle B, the bottom left blue circle C, and the bottom right red circle D.

- A can touch each of B, C, and D two times; 2+2+2 = 6.
- B has its intersections with A already counted but can touch C twice and D twice, so there are 2+2 = 4 more intersections.
- C can touch D 2 places.
- D has nothing to report – all the previous ones have touched it.

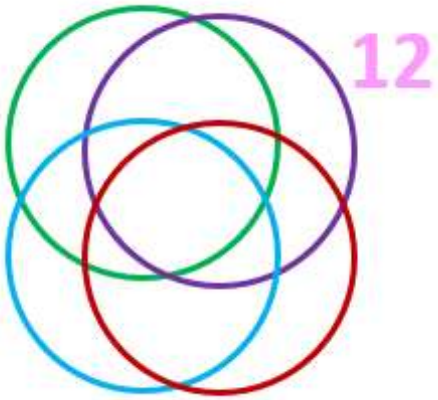
So we have 6 + 4 + 2 = 12 touch points at maximum.

Let's make a table to see what we have so far.

Number of circles	Upper bound of non-coincident intersections
0	0
1	0
2	2
3	0 + 2 + 4 = 6
4	0 + 2 + 4 + 6 = 12

Do you see a pattern? And does that pattern make logical sense? Have we seen numbers like this before?

Let's look at this logically



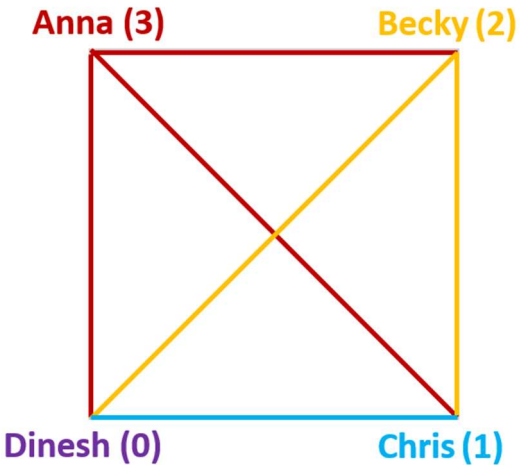
Let's say that you have a bunch of (non-coincident) circles like the four here. If you add another circle, are you convinced that you can make the new circle touch each previous circle twice? Take a moment if necessary and play with your compass to make some circle drawings to convince yourself that this is true.

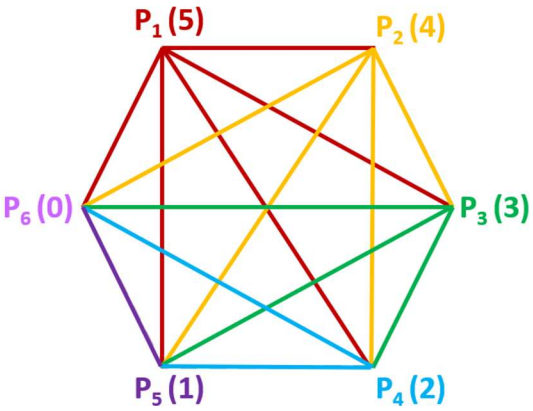
Okay, so let's look at the new circle. Let's say it's the 18th circle. We already have an answer for intersecting 17 circles and now we add 17+17 more, right?

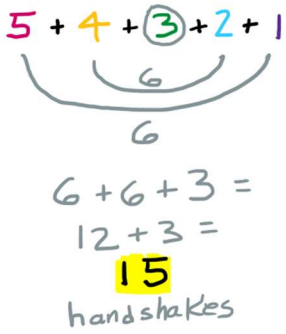
Let's build up our table a bit more and then see if we recognize a pattern that allows us to directly find the answer for any number of circles.

Do you know the Handshake Problem? If you have a bunch of people, how many handshakes would they make with each other is the question. How does this relate to our problem?

Number of circles	Upper bound of non-coincident intersections
0	0
1	0
2	2
3	0 + 2 + 4 = 6
4	0 + 2 + 4 + 6 = 12
5	0 + 2 + 4 + 6 + 8 = 20
6	0 + 2 + 4 + 6 + 8 + 10 = 30
7	0 + 2 + 4 + 6 + 8 + 10 + 12 = 42
8	0 + 2 + 4 + … + 12 + 14 = 56
9	0 + 2 + 4 + … + 14 + 16 = 72
10	0 + 2 + 4 + … + 16 + 18 = 90
n	??







Questions to consider

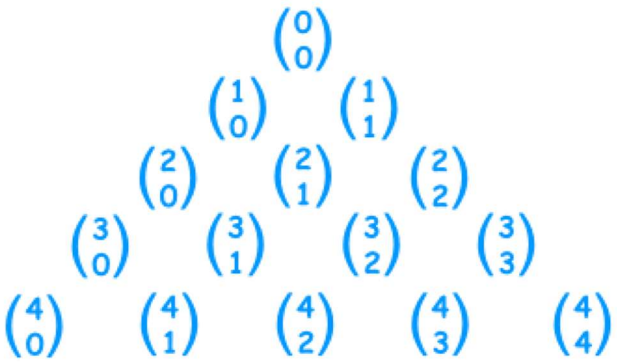
- Are there times in your everyday life when bounds are all you can use to predict something … like how long you will have to wait in line, how much traffic there will be, or something else?
- Be sure you understand the Handshake Problem. Is there an easy way to find the number of handshakes among n people? Can you treat the number of handshakes as an upper bound? Are all values 0, 1, 2, …, up to that bound possible?
- What does this have to do with the circle intersection problem?
- Can you easily predict the number of non-coincident circle intersections with n circles? Are all values 0, 1, 2, …, up to that value possible? Do you get different answers if you use different sizes for your circles?

If we have time, we'll discuss these questions in class. Happy almost April – see you next time!

Have Fun!
Mr. Barman

Notes for Parents

Circle intersections – bit of reminder/tutorial on combinatoric approach



I love counting problems like these. In algebra and beyond, your students may see these kinds of problems again but expressed through combinatoric notation. By way of reminder or quick brush introduction, nCk or $\binom{n}{k}$ or "n choose k" asks how many ways there are to select k things out of n when the order doesn't matter. In addition to combinatorics classes, you may have seen this in probability. Do you remember binomial coefficients? These numbers are also the terms of Pascal's Triangle[1]!

Briefly, let's look at the way of selecting, say, 2 co-leaders out of a group of 5. The first co-leader could be any of the 5 people and the second could be any of the remaining 4, so there are 5x4 ways. But if we select first Ravi and then Svetlana, that's the same as selecting Svetlana first and then Ravi[2], so we divide by 2! (so far I've justified 2 but 2! = 2x1 = 2; you'll see in a moment why I'm complicating things). So $\binom{5}{2} = \frac{5x4}{2!} = 10.$

How about selecting 3 co-leaders from 5? The quick answer is that this is equivalent to selecting 2 to **not** be co-leaders so it's the same answer as above; $\binom{5}{2} = \binom{5}{5-2} = 10.$

Another way of looking at this is that selecting 3 out of 5 where order doesn't matter means we have 5 x 4 x 3 choices but we have to divide out 3 x 2 x 1 because our 3 selected leaders can be in any order – ABC, ACB, BAC, BCA, CAB, or CBA. So $\binom{5}{3} = \frac{5\ x\ 4\ x\ 3}{3!}$ = 10.

In general, for k, n non-negative integers and k ≤ n, nCk = $\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!}$. This gives us the number of ways of selecting k (or n-k) things out of n things when order doesn't matter.

In the case of the overlapping circles problem, we want to find out the maximum number of places where n circles overlap for some non-negative integer n. Well there are $\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$ such pairs. Each pair can have two touch points so the upper bound of points where n circles touch is n(n-1), which corresponds to what we did though not using this kind of mathematics.

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[1] image from researchgate.net/figure/Binomial-coefficients-arranged-to-form-Pascals-triangle_fig1_286876476, accessed Feb. 28, 2023

[2] Order doesn't matter; if we draw A and then B, they are both co-equal leaders. But when order matters, such as when picking a president and a vice president, then A (president) followed by B (vice president) is very different than B (president) followed by A (vice president). In this case we work with a permutation not a combination; ${}_3P_2$ = 3x2 and, for n and k as described above, ${}_nP_k$ = n!/(n-k)!.

It's amazing that students come to terms with the circle intersections problem!

I teach this in my Stanford Math Circle and am so happy that the students seem to handle this intersections problem well. Almost everybody who has tried to answer this got it right and I bet that will be true today. I'm hoping that as they all think more about this for next week that they see that sums of even numbers is just twice sums of consecutive integers or Triangular Numbers. It's so gratifying for me to see students at such young ages being exposed and more or less understanding rather complex problems that I often pose to my prealgebra students, that of computing consecutive integers, consecutive even integers, and consecutive odd integers. These students will have a nice grounding when they see these patterns come up again some years down the road.

Student explorations

I hope that the children keep thinking about lower and upper bounds. They come up lots of places and it's comforting to know that even when circumstances can't give us exact answers, math has tools to help us quantify even these variable situations.

Hopefully they will realize that the size of the circle doesn't matter. Let me know if any of them come up with the idea of convexity – if you pinch a circle to make it concave, it can have an unbounded number of intersections as you pinch more and more.

Your child may intuitively see that all numbers up to the upper bound can be made in circle intersections. We demonstrated this for n=4 – but, inductively, wouldn't this be true for any larger integer n?

Most importantly, I hope that your child sees the straightforward relationships between Handshake Numbers and Triangle Numbers (T_n = 1+2+3+ … + n). Then Circle Intersection Numbers are just double.

Thanks for coming to Chapel Hill Math Circle. See you in two weeks.

Till next time!

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