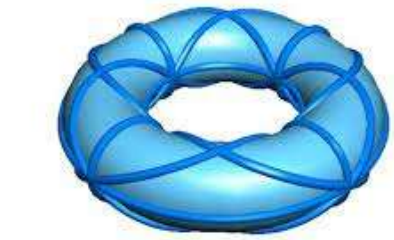


Chapel Hill Math Circle
Session 15 – April 26, 2025:
Square Numbers
Beginners' Group (grades 1-3), 10:30-11:30a
Mr. Barman – dilip@trianglemathinstitute.com

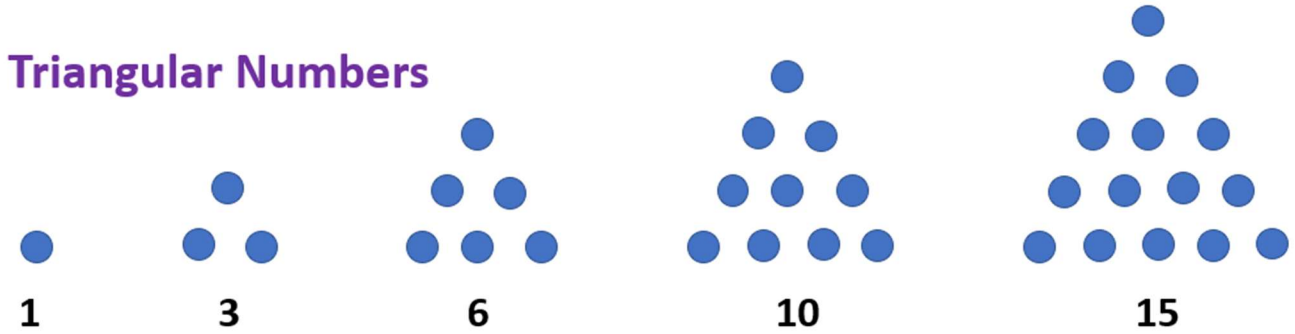
Supplies needed per student: graph paper, scissors, colored pencils, as well as paper and pencil

Welcome to Math Circle session 15. It's our last of the school year. I also wanted to take a moment to thank Dr. Linda Green for organizing for some years our math circle. Some of you may know that she is moving to another job and, while math circle will continue next year, she will not be part of it. If you see her in the halls, please thank her for all that she's done for forming and running the Chapel Hill Math Circle all of these years!

Let's look at numbers which form shapes. We've talked about Triangle (or Triangular) Numbers before; today let's look at Square Numbers.

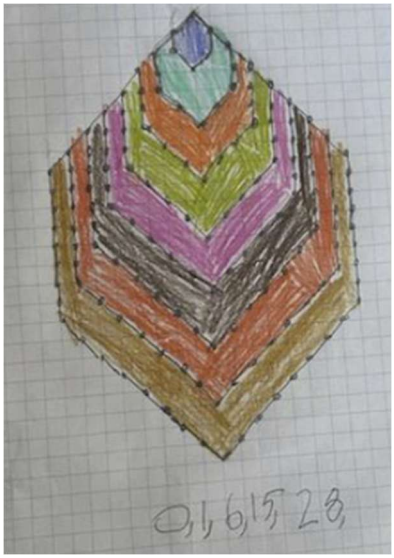
Figurate numbers

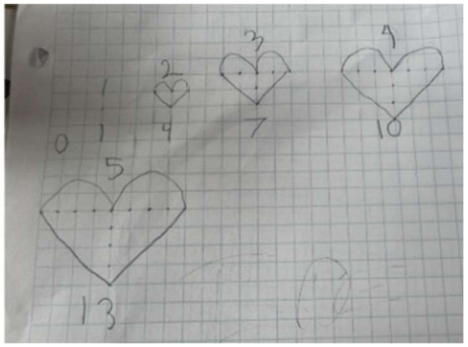
Figurate numbers form shapes when you express them as dots. Triangle Numbers come up fairly often and you may have seen them in math circle.

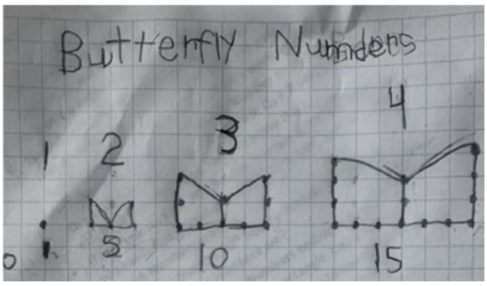


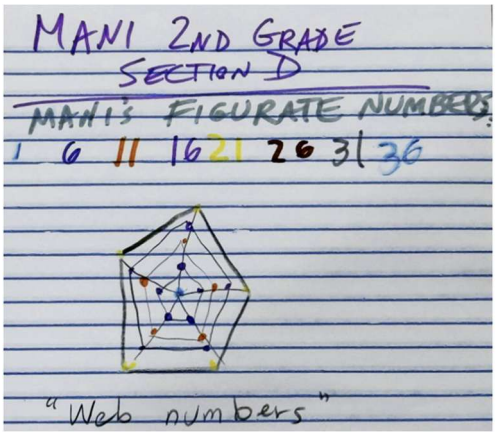
Make up your own figurate numbers. Use a shape that you like and name your set of numbers! Don't yet look at the next page but there I'll share some examples that students in my Stanford Math Circle have come up with to maybe inspire you to keep inventing. Extra: how can you find the 5^{th}, 10^{th}, 100^{th}, and n^{th} numbers in your shapes where n is some counting number?

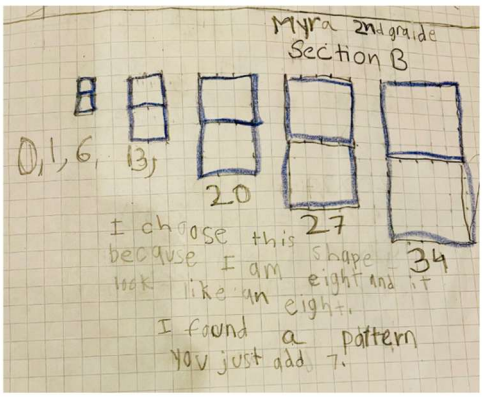
Here are some figurate numbers from my Stanford Math Circle 1st and 2nd graders. The first is by 1st grader Everett from April 2025 of Hexagonal Numbers. Twin 1st grade sisters Roya (Roya Heart Numbers) and Mina (Mina Butterfly Numbers) shared also in April 2025, as did 2nd grader Mani with his Web Numbers (pentagons that reminded him of spider webs). I had a student in 2023 named Myra; then a 2nd grader, she made numbers based on her age of 8.







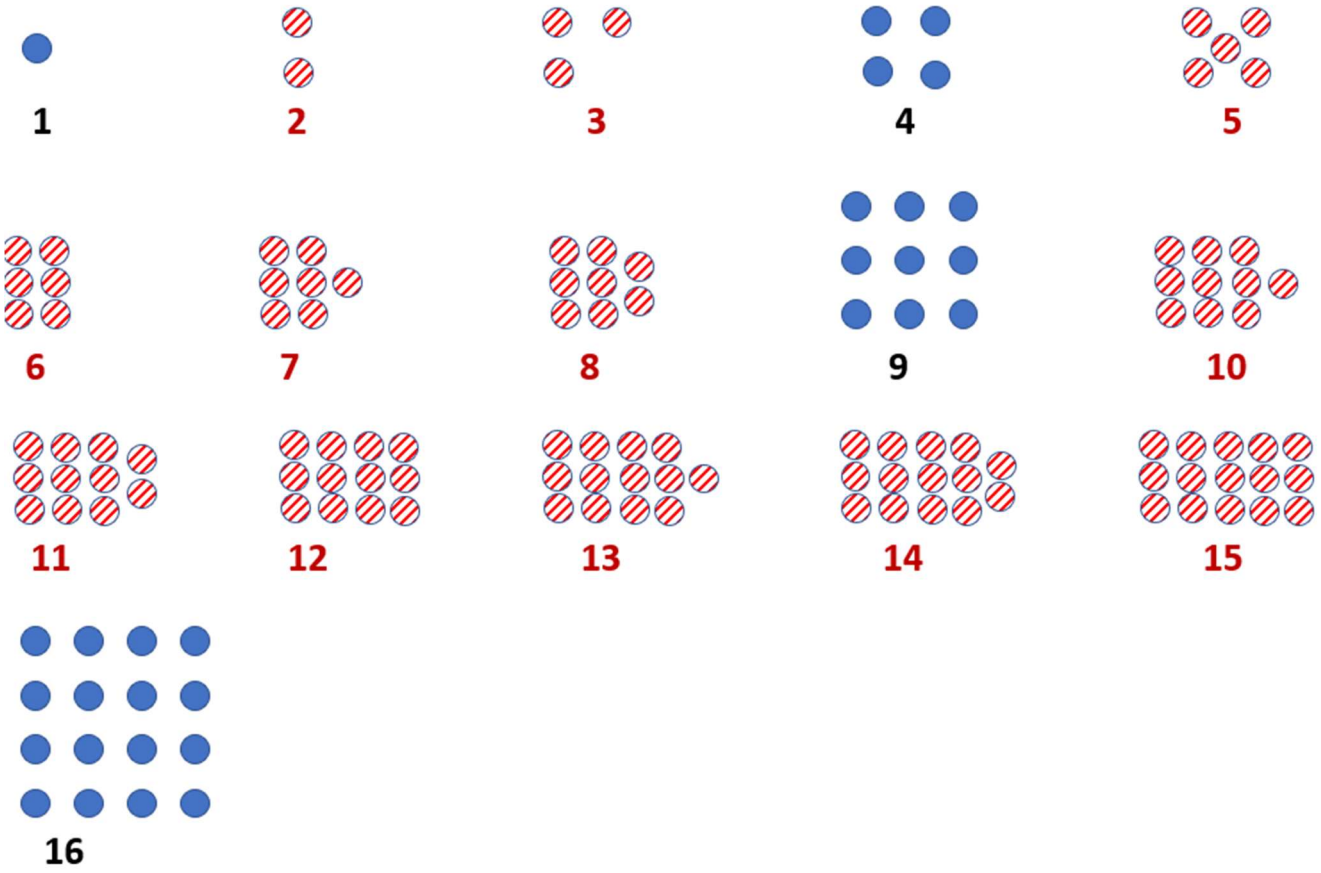




Squares from dots

Can you make squares from dots? Can you arrange 4 dots into a square? What's the smallest square that you can make in this way? How many dots can you use to make a square? Which numbers can be made by arranging dots in a square? Try it!

Did you get something like this?



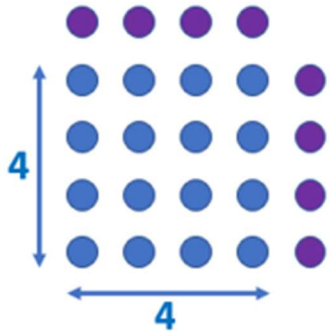
So the first few Square Numbers are 1, 4, 9, and 16? Can we look at a "base case" and throw in 0 too? It's like a ghost square! A single dot looks like a dot that wants to become a square when she grows up – we call 1 a "degenerate" square. So our first few Square Numbers are:

0, 1, 4, 9, and 16

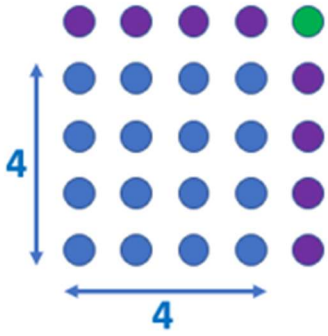
… and does your "intuition" tell you that the next Square Number is not in this row? How? *Hint: can you think of a way to build the next square given one square? How many dots are needed?*

We need 4+4+1 dots to get the next Square Number after 16

The Square Number 16 has 4 dots per side



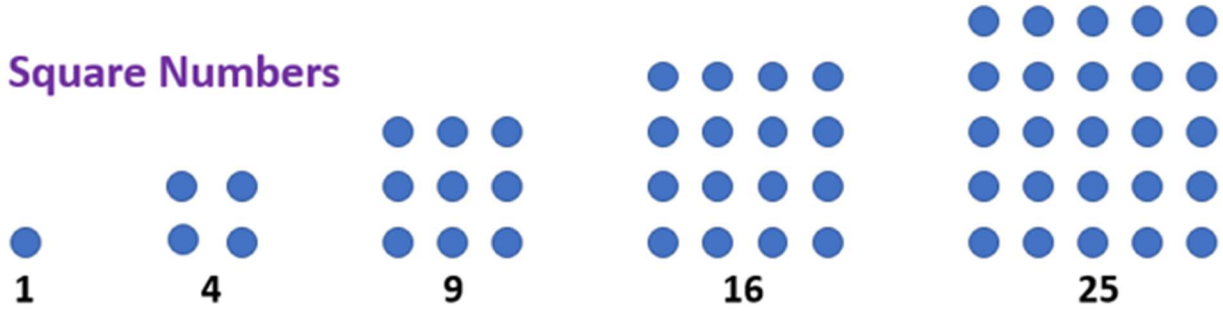
Add in 4 on top and 4 to the right …



… and 1 more to make the next square

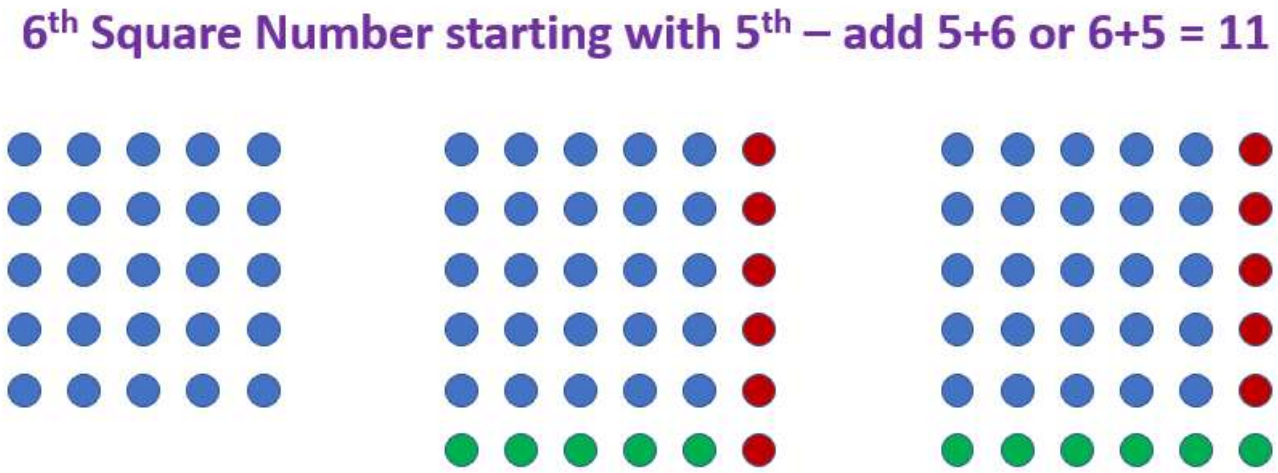
What is the pattern? If I tell you that the 10^{th} Square Number is 100 (10 10s = 100), can you easily find the 11^{th} Square Number? How about the 12^{th} and 13^{th}? If I tell you that the 30^{th} Square Number is 900 (30 30s; 10 30s is 300 so 30 30s is 300+300+300), how can you figure out the 29^{th} and 28^{th} ones?

How would you figure out the next few Square Numbers?



Wouldn't the next one be 6+6+6+6+6+6? That's 12+12+12 = 24+12 = 36.

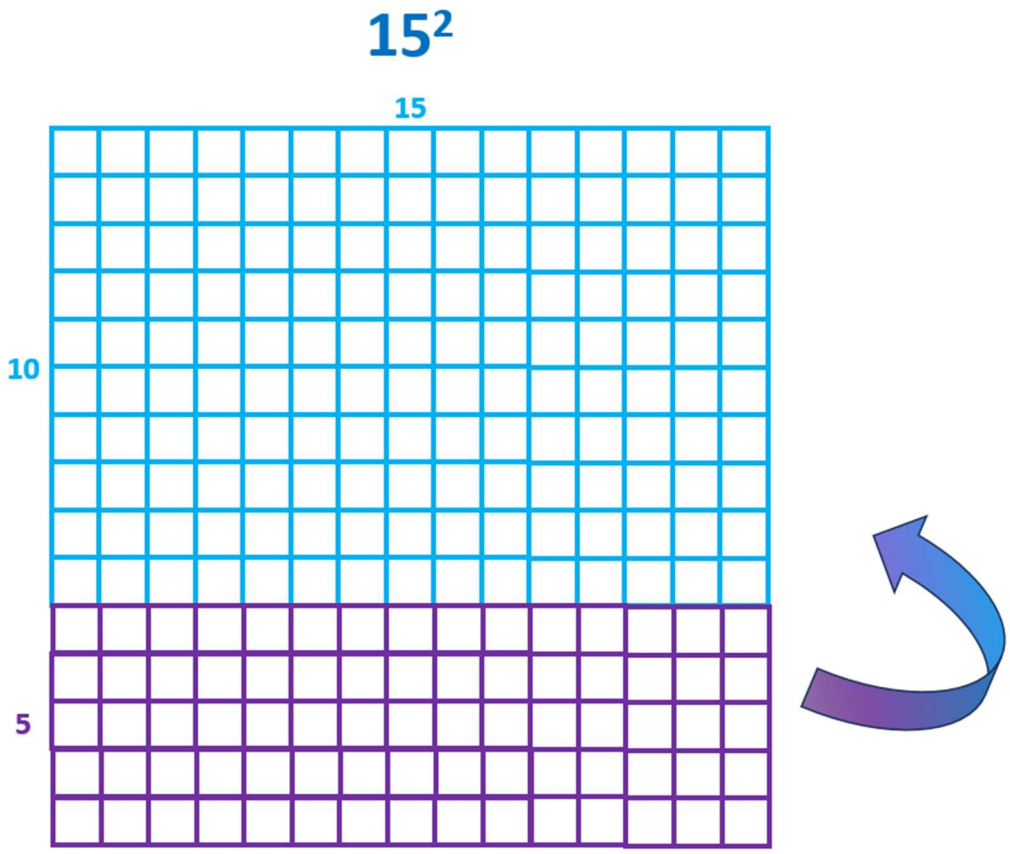
Think of it as building a room. We can go from one square room, say a 5x5, and make it the next bigger one (here, 6x6) by adding 5+5+1, or just adding 5 and then 6, right? If we know the 5^{th} Square Number, we get the next one by adding a row of 6 on top and a column of 5 to the right – or a column of 6 to the right and a row of 5 atop. Or we add a column and a row of 5, and then add a dot.



There's a special trick for easily finding Square Numbers for numbers that end in 5 like 15, 25, 35, …, 95, 105, 115, …, 2005, 2010, …. Take a piece of graph paper and make a 15x15 square. You're smart and can skip count by 15 – let's try! 15-30-45-60-75-90-105-120-135-150 [hey! I know 10 15s is 150 so I could have started here!] -165-180-195-210-225. 15 15s is 225. But I am much faster skip counting by 10s.

Take a few minutes and use scissors to cut your 15x15 square into easier to count shapes. Talk with a friend and come up with a way to rearrange the 15x15 to make it fast to count. If you find a way, try applying it to a 25x25 square. Does your way easily generalize to any square with side length ending in 5?

Let's make some diagrams. Did I mention that we sometimes write a little number 2 up in the air to indicate a Square Number? For example, □$_3$ = 3 3s = 9 is also written as $3^2 = 9$. Let's look at 15^2. By the way, this is from material that I created to teach my 3rd grade class.

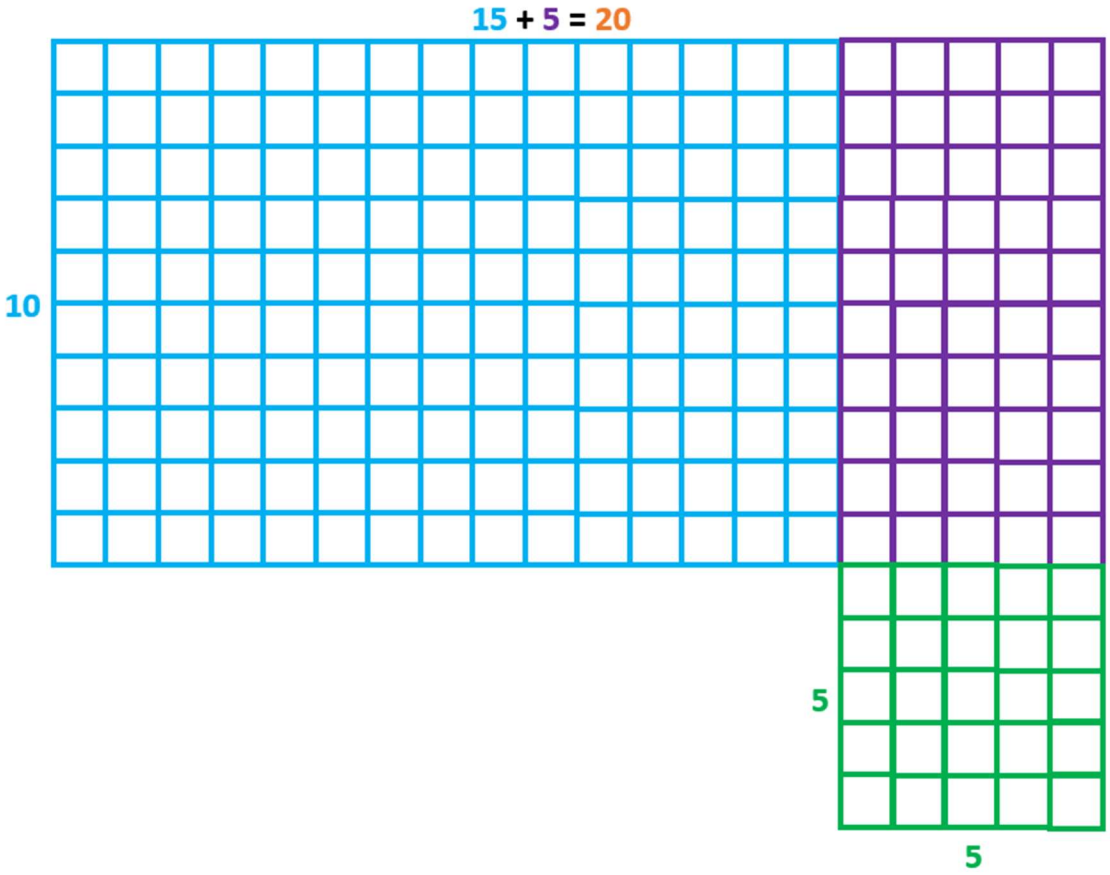


It's easier to skip count by 10 than by 15 so let's cut out the bottom 5 rows. In a moment we'll flip that rectangle and move it to the right.

Once we move the bottom 5 rows, can we see that the 15th Square Number is 10 20s plus the green "tail" of 5 5s, or 200 + 25, 225.

This will work for any number ending in 5, won't it? See the diagram on the next page. For example, 25^2 = (20 30s) + 25; 10 30s is 300 so 20 30s is 600 and 25^2 is 625. See if you can find 35^2, 45^2, 55^2, 95^2, and 105^2.

$$15^2 = (10\ 20\text{s}) + 5^2$$



Have a great summer!

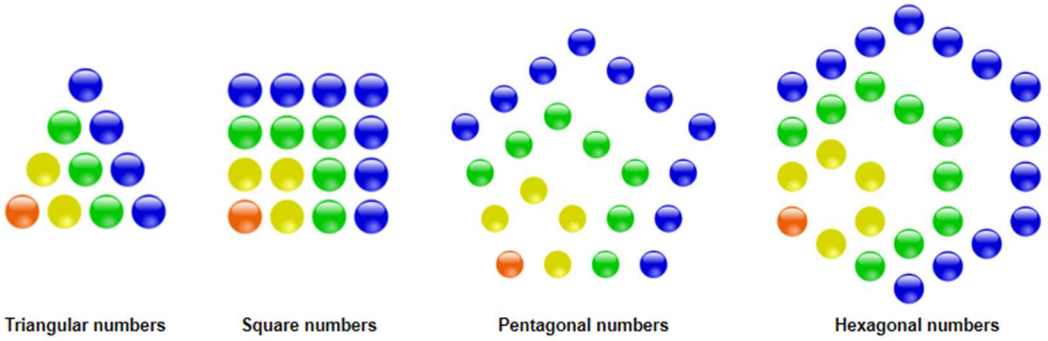
I know summer is still almost two months away and even most school summer breaks are over a month away. In any case, today is our last math circle of the school year. I love working with you and will miss you. If you're a current 1st or 2nd grader, I hope that I'll see you again next year when we start up again. I hope that 3rd graders continue on with our Elementary Math Circle. And I hope that I'll get to work with all of you perhaps in my math classes or, when you are rising 5th graders, my camps at NC School of Science and Math.

Enjoy
Mr. Barman

P.S. For some fun math activities, have your parents check out trianglemathinstitute.com/math and trianglemathinstitute.com/mathtips .

Notes for parents

Figurate numbers[1] are numbers that can be built on regular geometric arrangements of dots, like Square and Triangular Numbers. There isn't one canonical definition of "figurate" – and it's nice here to go with an open-ended invitation to play and define your own definition, as I'm asking the children to do for homework. A good reference that you and your child may enjoy is a page by Mathigon[2].



In today's lesson I hope that children enjoyed seeing some basic patterns in Square Numbers. I am a firm believer in sticking with addition for 1st and 2nd graders (I know that we have 3rd graders as well in our math circle). Many of them may have been exposed to multiplication – some may even claim to know exponentiation, negative numbers, algebra, and more. There's so much to probe with just addition that I'd rather stick to using 2+2+2+2 instead of 2x4 and making sure that students really understand concepts before progressing.

Square Numbers are one way for them to be exposed to exponentiation, leapfrogging multiplication even. In third or fourth grade I expect students to become familiar with Square Numbers and recognize the first twenty or twenty-five of them. With the background they are getting here this familiarity will be more tangible than squaring numbers; they should get a more innate understanding of their structure and how numbers grow from perfect squares.

I may informally also discuss Rectangular Numbers but as I put the lesson together I realized that I have enough material for a leisurely math circle and didn't want to rush things. The idea is simple and I hope to present it in the fall. I believe that this exposure will yield good dividends. In a few years when these children are taught about prime numbers (I teach it in 4th grade), they may recall Rectangular Numbers. I encourage you to remind them if they don't themselves remember, and ask them to think about the geometric interpretation of prime numbers.

[1] This graphic of examples of a particular kind of figurate number, polygonal number, is from Stefan Friedrich Birkner shared with a Creative Commons Attribution-ShareAlike 3.0 license as cited at oeis.org/wiki/Figurate_numbers, accessed Feb. 14, 2023.

[2] mathigon.org/course/sequences/figurate

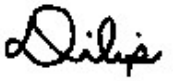
Pronic Numbers are Rectangular Numbers that have side lengths that differ by 1. A 1x2 or 2x3 or 3x4 rectangle is an example. Pronic and Triangular Numbers have a nice relationship that I encourage to explore with your child.

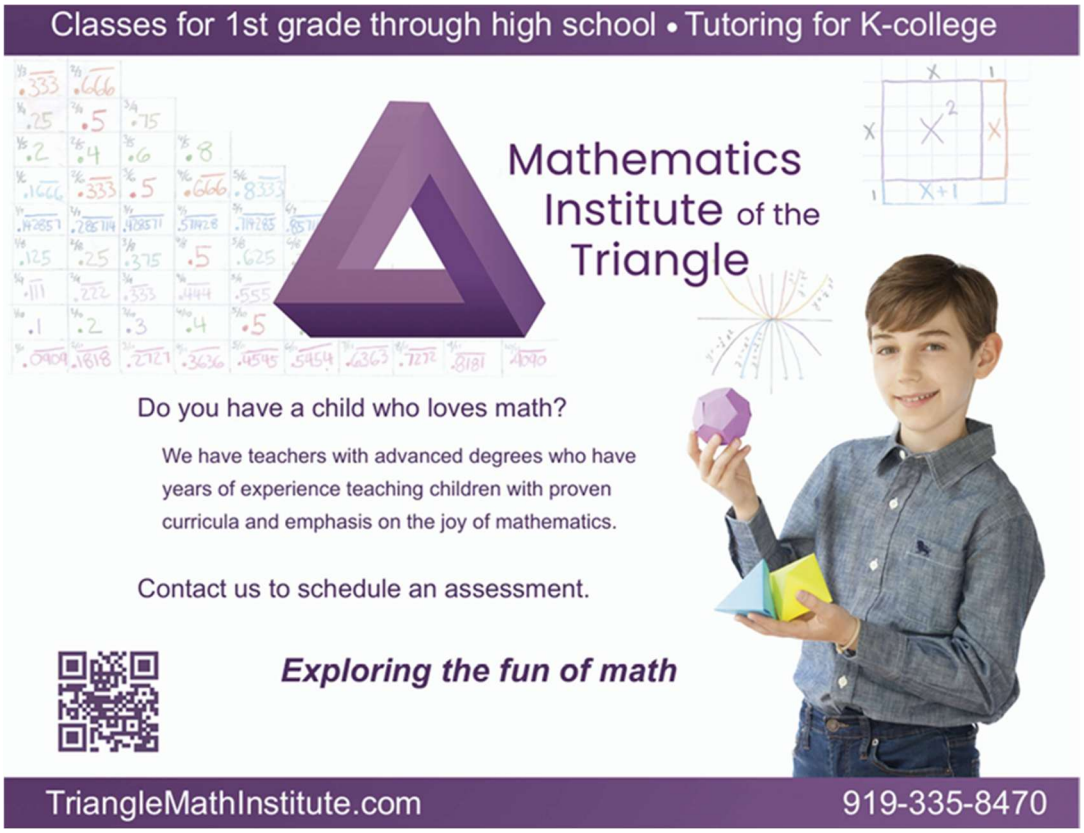
Fall 2025: start of fourth year at Mathematics Institute of the Triangle

I love teaching math so much that I helped to open an afterschool/weekend math enrichment school here in Chapel Hill a few years ago. This fall will be our fourth year and soon we'll begin registration. We definitely will be offering classes for 2nd grade through high school and plenty of 1-on-1 tutoring. We are likely offering a 1st grade class and a few daytime homeschool classes. Please be in touch for more information and to possibly schedule a placement assessment and school tour. Did I mention that we are also thinking of offering a language arts and maybe a science class?

Thanks for participating in the Chapel Hill Math Circle. Have a good summer!

Till next time!



Dilip Barman dilip@trianglemathinstitute.com

P.S. Check out some fun math circle-inspired and other activities I recently shared at trianglemathinstitute.com/math . You may have already seen trianglemathinstitute.com/mathtips .