Chapel Hill Math Circle
Session 4 – October 11, 2025: Making Change, Repeated Subtraction
Beginners' Group (grades 1-3), 10:30-11:30a
Mr. Barman – dilip@trianglemathinstitute.com
Supplies needed: coins (real or representative) – 4 quarters, 10 dimes, 20 nickels, and 100 pennies per student

Welcome to Chapel Hill Math Circle! Let's look at coins and subtraction today.

Counting Coins: Quarters

Four quarters make a dollar. How many dollars is 8 quarters? 20 quarters?

Banks organize quarters into rolls that are worth $10. How many quarters would go in a roll? I use quarters for Halloween[1]; how many rolls should I get if I expect 50-70 trick or treaters?

[1] The picture here is of my then almost-7-year-old 1st grade daughter Anuragini on Halloween 2015 at 8:21p. Boo!

Making change

If you have a dollar bill, you might "break" it into 4 quarters. Or you might "break" a $5 bill into maybe 3 $1 bills, 4 quarters, and 10 dimes. Let's be sure to discuss why we make change. Do you have any ideas?

Do you know that $1 = 4 quarters = 10 dimes = 20 nickels = 100 pennies? Let's say that you have an uncle who wants to give you a dollar ($1 which equals 100 pennies which equals 100¢). He can, for example, give you a dollar bill ($1 = 100 pennies), four quarters (25¢ + 25¢ + 25¢ + 25¢), 10 dimes (10¢ + 10¢ + 10¢ + 10¢ + 10¢ + 10¢ + 10¢ + 10¢ + 10¢ + 10¢), 20 nickels (each worth 5¢), or 100 pennies (each worth 1¢).

But there's lots of other possibilities. What if he gives you 3 quarters, 2 dimes, and 5 pennies? Let's try counting some of the possibilities.

- Using just quarters and dimes, how many ways can he give you $1?
- Using just quarters and nickels, how many ways can he pay you $1?
- Using just quarters and pennies, how many ways can he pay you $1?

Let's take a few minutes and try these problems. Can you solve them with any shortcuts?

QD:

QN:

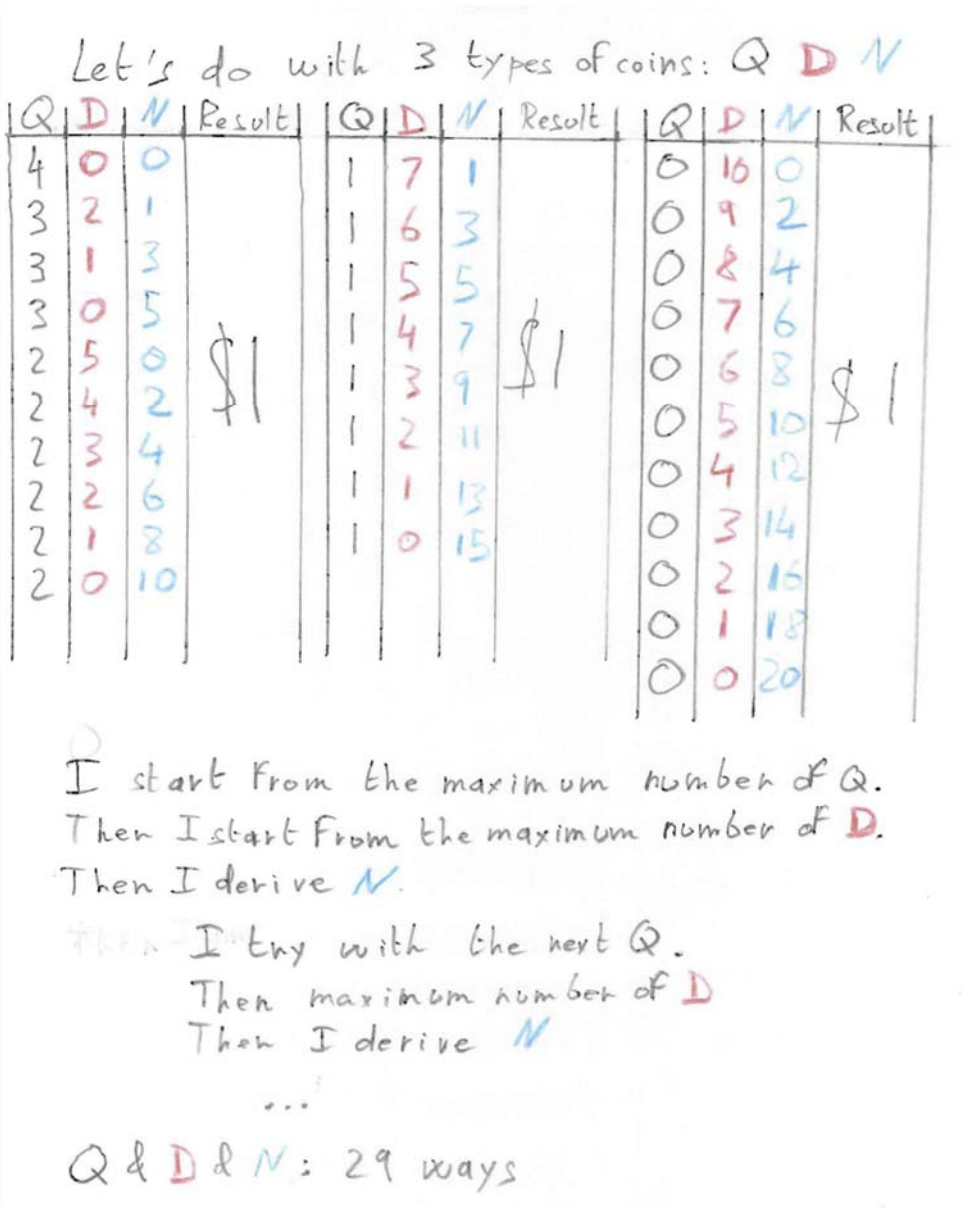
QP:

Take a few minutes to figure out how many ways there are to make a dollar with quarters, dimes and nickels. Use a piece of scratch paper and make three columns, one for Q (quarters), the next for D (dimes), and the last for nickels (N). Be sure to count in some order. Write your answer to the right.

Did you get 29 ways? Here is how my Stanford Math Circle 2nd grade student Luc did it in October 2025.

Here are a few things to think about.

- How did he order his work? Is there any other way that the work could be ordered?
- What patterns do you see?



Let's look at one pattern that I see. How many ways are there to make $1 with 4 quarters? One way. With 3 quarters? Three ways. With 2 quarters? Keep going and you will see that we have 1 + 3 + 6 + 8 + 11 = 4 + 14 + 11 = 18 + 11 = 29 ways. What pattern is 1, 3, 6, 8, 11?

At home try this if you have a half hour or so sometime: count the number of ways to make $1 with quarters, dimes, nickels, and pennies. Do you have any guesses about how many ways there are?

Quarters	Dimes	Nickels	Pennies
4 (0¢)			
3 (25¢)	2 (5¢)	1	
	2 (5¢)	0 (5¢)	5
	1 (15¢)	3	
	1 (15¢)	2 (5¢)	5
	1 (15¢)	1 (10¢)	10
	1 (15¢)	0 (15¢)	15
	0 (25¢)	5	
	0 (25¢)	4 (5¢)	5
	0 (25¢)	3 (10¢)	10
	0 (25¢)	2 (15¢)	15
	0 (25¢)	1 (20¢)	20
	0 (25¢)	0 (25¢)	25

If you do this problem, look for patterns. I started and got this table to the left. I noticed that we can make a dollar:

- 1 way with 4 quarters
- 2+4+6 ways with 3 quarters

Keep going!

Subtracting 3s and 5s from 20 – what can you get?

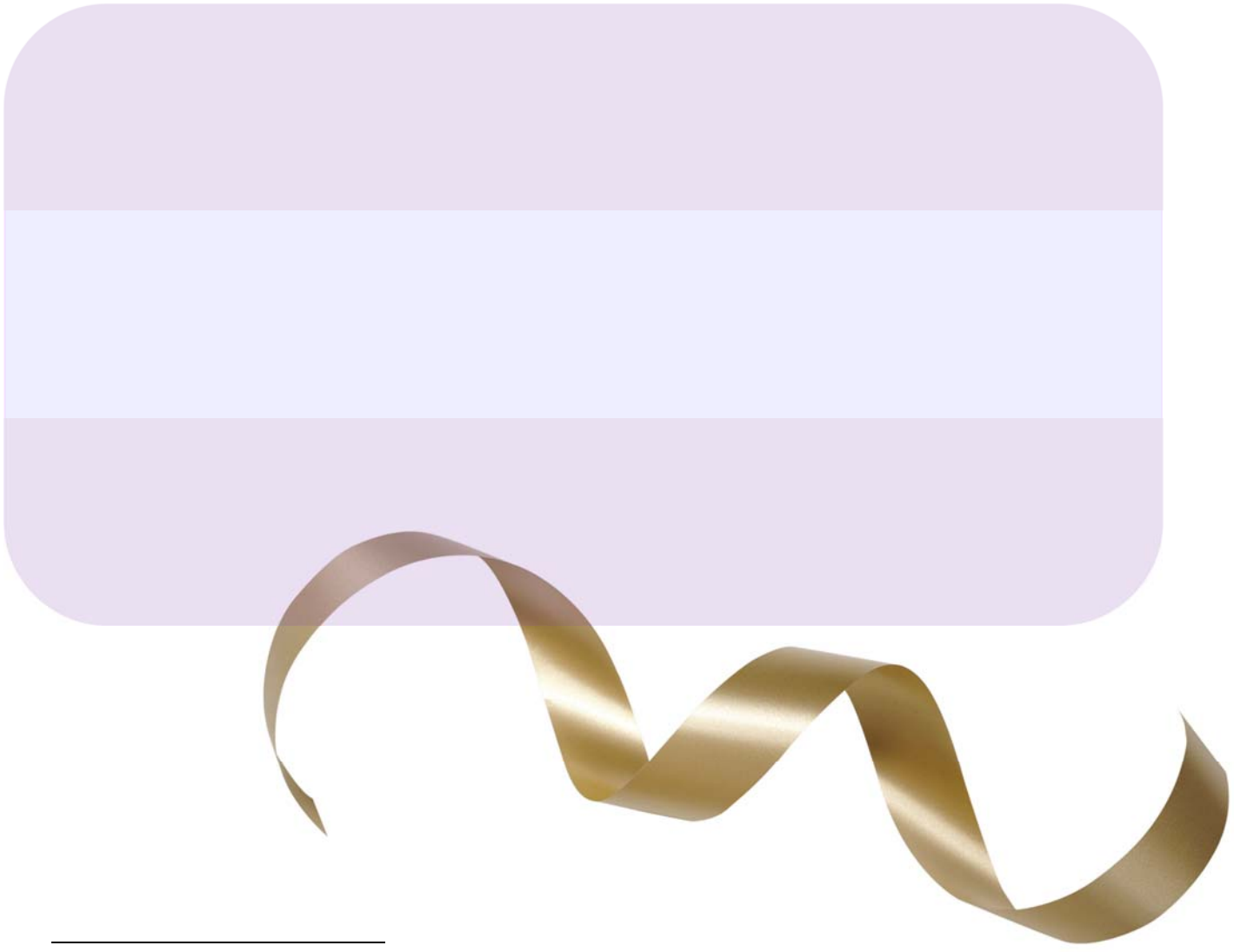
Let's conclude with playing with subtraction. What can I get[2] starting at 10 if I can subtract any number of 4s and 7s? I can get:

- 10 - 7 = 3
- 10 - 4 = 6
- 10 – 4 – 4 = 2
- 7 – 4 = 3 (again)

And that's it. I can get 2, 3, and 6.

Please try this problem: what can I get, starting at 20, and subtracting any number of 3s and 5s? Take a few minutes to try this.

I used a "brute force" method for the problem above. Can you think of any "elegant" ways?



[2] If you know about negative numbers, we are looking for only non-negative ones: 0, 1, 2, 3, ….

Inspired by my 2nd grade student Anika and a problem we sometimes look at in math circle, the Frobenius Coin Problem, here is one way to solve it. Let's start at 20 and count backwards. I'm going to count in rows of 5. Why 5? Because if I can make a number, say 12 (3+3+3+3), I can make everything under it by subtracting more 5s. (I could have made a grid that was 3 wide as well – do you agree?)

- Subtracting 5s, I can get to 15. That means that I can also get to 10 and 5.
- Okay, now let's subtract 3 from 20. I get to 17 – and can definitely get to 12, 7, and 2 by subtracting 5s.
- Let's continue subtracting 3s and get to 14 (and 9 and 4 under it), 11 (and 6 and 1), and 8 (and 3)

What's left are the numbers I can't make – 20, 19, 18, 16, and 13.

Another of my students, 2nd grader Luc, used a similar method in October 2025 that he called a grid. My 1st grade student Reyansh, also in October 2025, used a "tree"; he subtracted 3s to the left and 5s to the right. I am lucky to get such smart and creative students – like you!

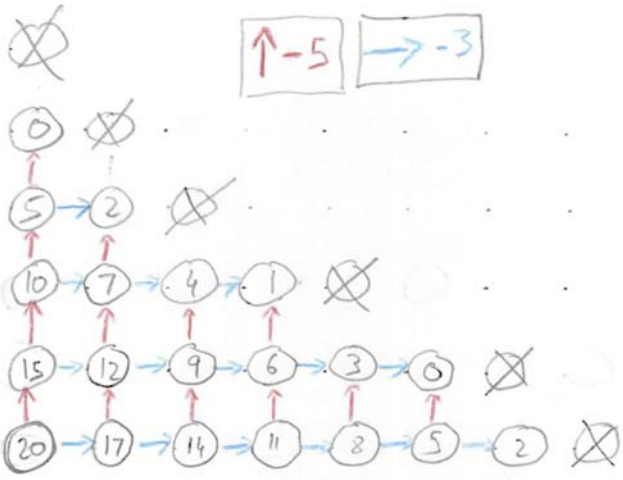
I hope you had fun. Try the QDNP change problem if you can. See you next time!

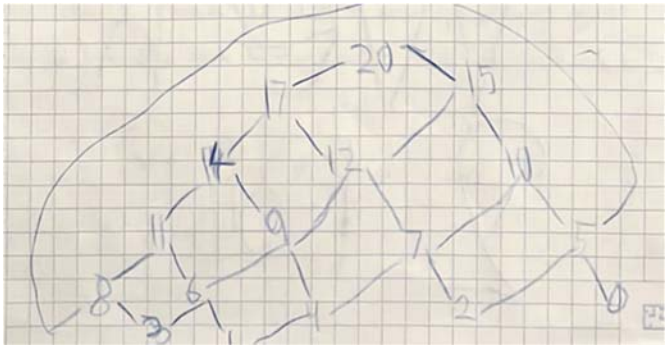
20	19	18	17	16
15	14	13	12	11
10	9	8	7	6
5	4	3	2	1

20	19	18	17	16
15	14	13	12	11
10	9	8	7	6
5	4	3	2	1

20	19	18	17	16
15	14	13	12	11
10	9	8	7	6
5	4	3	2	1

20	19	18	17	16
15	14	13	12	11
10	9	8	7	6
5	4	3	2	1





Have Fun!
Mr. Barman

Notes for parents

Counting – change, ways to go from one intesection to another, diagonals in a polygon, so much more – can be a very interesting problem. The first step is to understand the problem, maybe solve a small version of the problem, and then look for patterns. Patterns are the heart of mathematics.

For the "QDNP" problem (making change for a dollar with quarters, dimes, nickels, and pennies), I've had a few 1st and 2nd graders solve this by brute force – successfully! I encourage you to work with your child to start this exhaustive effort and then stop and look for the shortcut that I began in the student notes. I'll include how it works out below.

Till next time!

Dilip

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We have some patterns and are getting there!

So we have (1) way with 4 quarters, (2+4+6) ways with 3 quarters, and (1+3+5+7+9+11) ways with 2 quarters. Let's move on to 1 quarter. The most dimes we can use with 1 quarter is 7 dimes; 100¢ - 25¢ = 75¢ and 7 dimes makes 70¢. We still need 5¢ which has 2 solutions, 1N or 5P. Then we look at 6 dimes; we must make up 15¢. That can be done with 3, 2, 1, or 0 nickels, so there are 4 solutions. You got it – with 5 dimes there are 6 solutions, with 4 dimes there are 8 solutions, with 3 dimes there are 10 solutions, with 2 dimes there are 12 solutions, with 1 dime 14, and with 0 dimes 16, so **1Q** has 2+4+6+ … + 12+14+16 more solutions.

How about no quarters? We can have 10 dimes – there's only one way to do that. How about 9 dimes? There are 3 ways to cover the remaining 10¢ (choice of 2, 1, or 0 nickels with the rest covered by pennies); 8 dimes with 5 ways (4, 3, 2, 1, or 0 nickels with the rest covered by pennies) the remaining 20¢; 7 ways with 7 dimes; 9 ways with 6 dimes; 11 ways with 5 dimes; 13 ways with 4 dimes; …; 21 with 0 dimes.

Finally, we can add it all up!

The number of solutions of making change from a dollar with quarters, dimes, nickels, and/or pennies is:

- 1 *with 4 quarters*
- 2+4+6 *with 3 quarters*
- 1+3+5+7+9+11 *with 2 quarters*
- 2+4+6+8+10+12+14+16 *with 1 quarter,* and
- 1+3+5+7+9+11+13+15+17+19+21 *with no quarters*

So many patterns! What patterns do you see?

Let's add it up; we have:

(1) + (2+4+6) + (1+3+5+7+9+11) + (2+4+6+8+10+12+14+16) + (1+3+5+7+9+11+13+15+17+19+21)
= (1) + (2+4+6) + (1+2+3+4+5+6+7+8+9+10+11+12) + (14+16) + (1+3+5+ … + 21)
= 13 + 78 + 30 + 121 = 13 + 30 + 199 = 43 + 199 = 42 + 200 = **242**